The application of tensor network method in three dimensional quantum system

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Motivation

- The classification of 3d bosonic topological order (TO)& symmetry protected topological order(SPT) is well known (**fixed point wave function**)
- We would like to study quantum system (with topological order) in 3D

  But How to detect those topological order phase numerically?

  Numerical tool: 3D HOTRG , 3D CTM,…
- To simplify our problem, we will consider **fixed point wave function with deformation** (not from Hamiltonian)

Outline

* **Introduction :**

  topological order

  2D and 3D $\mathbb{Z}_N$ toric code

* **Numerical method:**

  Tensor-Network scheme for modular *S* and *T* matrices (tnST)

  3D high order tensor renormalization group

* **Numerical results:**

  Case study: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ topological order in 3D

  Dimensional Reduction to 2D

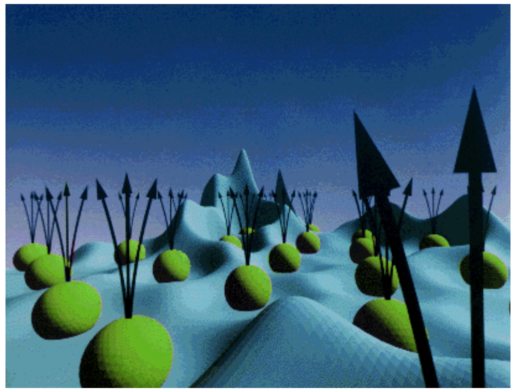
  3D AKLT (symmetry) state and deformation

* **Summary**

Introduction: Topological order

* **Beyond Landau (symmetry-breaking) paradigm**
  eg. Fractional Quantum Hall, Spin Liquid, ...

[Tsui,Stomer,Gossard '82,Laughlin '83, Anderson '73,...]



* **Topological order characterized by:**

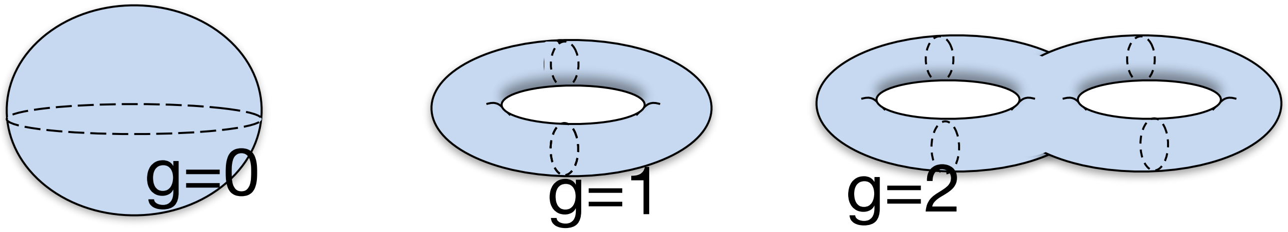
  Topology-dependent ground-state degeneracy (N^g)

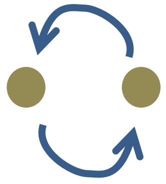
  Nontrivial excitations and statistics (usually in 2d)

  Long-range entanglement [Wen '90]

* **Potential application in fault-tolerant quantum computation**

[Wen and Niu '90]

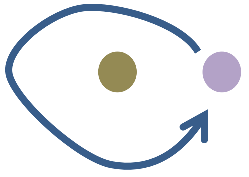




$|\Psi\rangle \rightarrow e^{i\varphi}|\Psi\rangle$ anyon

$|\Psi\rangle \rightarrow |\Psi\rangle$ boson

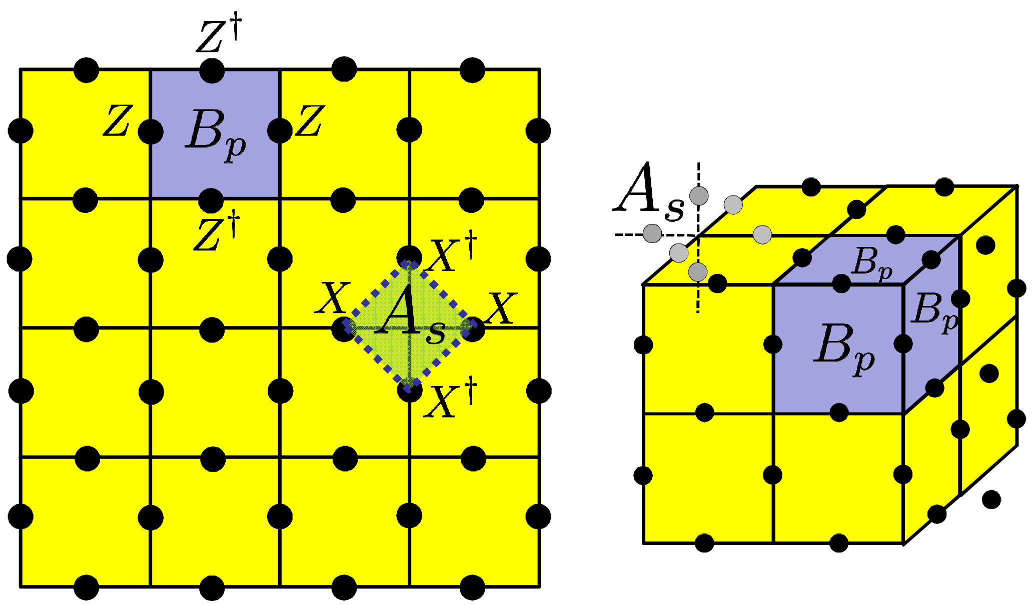
$|\Psi\rangle \rightarrow -|\Psi\rangle$ fermion



Topological order: $\mathbb{Z}_N$ Toric code

* 2D and 3D: **spins reside on edges**

  N -state degrees of freedom located on the link $|q\rangle_i$



* **The Hamiltonian** of the $\mathbb{Z}_N$ toric code

$$H = -\frac{J_e}{2}\sum_s (A_s + A_s^\dagger) - \frac{J_m}{2}\sum_s (B_p + B_p^\dagger)$$

* **The operators** Z_i and X_i as

$$Z_i|q\rangle_i = \omega^q|q\rangle_i; \quad X_i|q\rangle_i = |q-1\rangle_i; \quad \omega = 2e^{2\pi i/N}$$

* **Ground state satisfy**

$$A_s|G.S.\rangle = B_p|G.S.\rangle = |G.S.\rangle$$

Topological order: $\mathbb{Z}_N$ Toric code

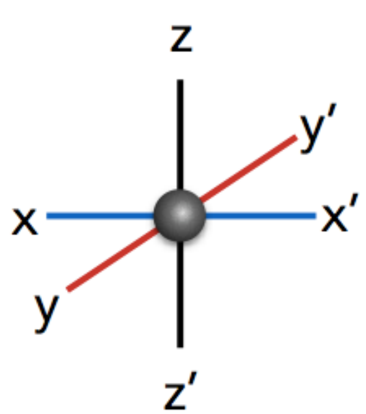
- **Degeneracy on 2,3-torus**

  2D: $\#_{deg} = N^2$ 3D: $\#_{deg} = N^3$

- **Representative ground states can be written as a tensor network**

$$|\psi\rangle = \sum_{s_i} tTr(\bigotimes_v P \bigotimes_l G^{s_i})|s_1, s_2, ...\rangle,$$

@ each site:p



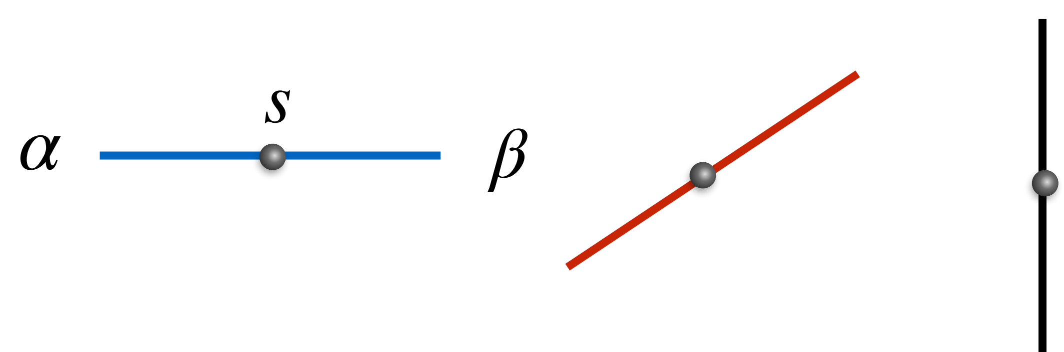
$P_{xx'yy'zz'} = 1$

only if

$x - x' + y - y' + z - z' = 0 \; (mod \; n)$

@ each link (3 direction)

$G^s_{\alpha,\beta} = \delta_{s,\alpha}\delta_{s,\beta}$



→ **Deform toric** $G^s_{\alpha,\beta} = f_s \, \delta_{s,\alpha}\delta_{s,\beta}$

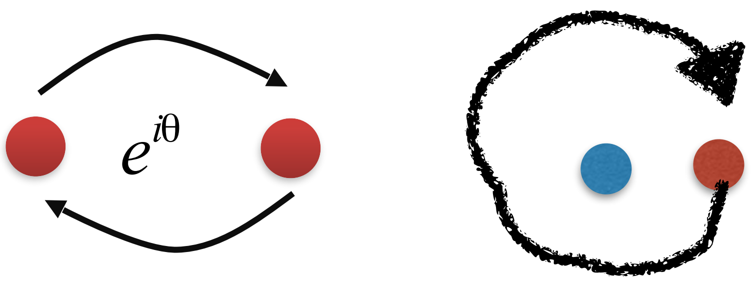
- Ground state:

  → use the string operater to get other ground state

  e.g. 2d TC $|\psi_{\alpha,\beta}\rangle = (\mathcal{Z}_1)^\alpha(\mathcal{Z}_2)^\beta|\psi_{0,0}\rangle$

Order parameter: from wave function overlap

* Topological order characterized by its **quasiparticle excitations-** anyons (with nontrivial braiding statistics)

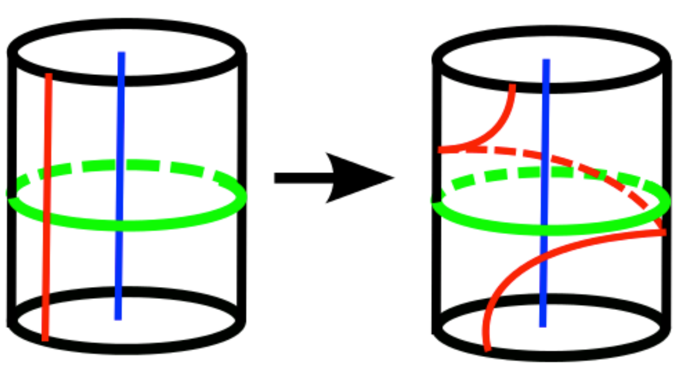


* Mathematically, the **braiding statistics** is encoded in the modular matrices.

* The modular matrices, or S and T matrices, are generated respectively by **the 90º rotation** and **Dehn twist** on torus.

$$\langle\psi_a|\hat{S}|\psi_b\rangle = e^{-\alpha_S V + \mathfrak{o}(1/V)} S_{ab}$$
$$\langle\psi_a|\hat{T}|\psi_b\rangle = e^{-\alpha_T V + \mathfrak{o}(1/V)} T_{ab},$$

$\{|\psi_a\rangle\}_{a=1}^N$ **:degenerate ground state**



[Hung & Wen '14; Moradi & Wen '14]

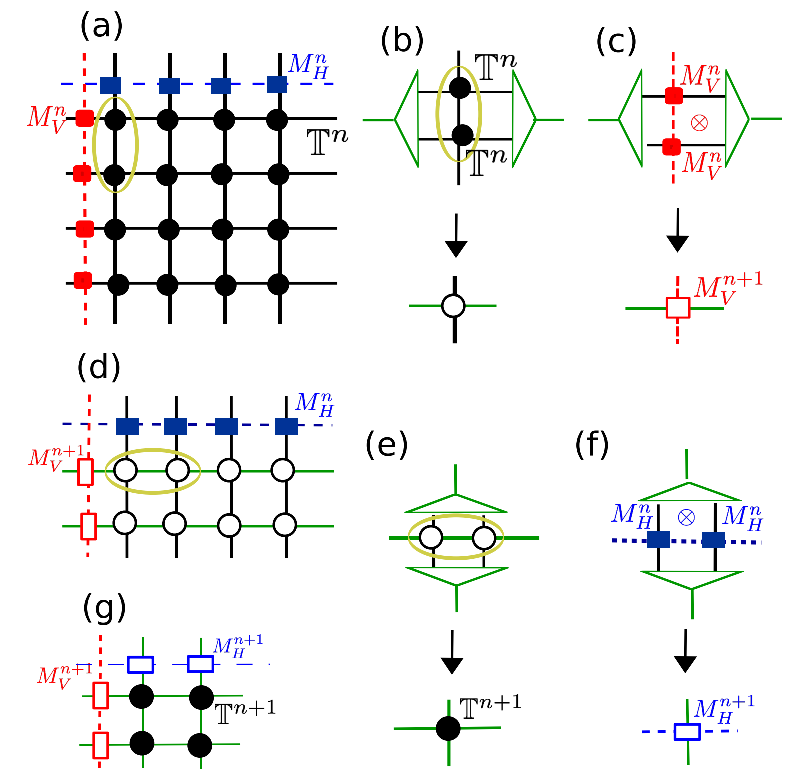
Previous work: 2D topological order with deformation

- Start from a wave function in 2D with deformation
  ⇒ By tuning a parameter to study the phase transition

$$\langle\psi_a|\hat{S}|\psi_b\rangle = e^{-\alpha_S V + \mathfrak{o}(1/V)} S_{ab}$$
$$\langle\psi_a|\hat{T}|\psi_b\rangle = e^{-\alpha_T V + \mathfrak{o}(1/V)} T_{ab},$$

We propose a way -tnST **"Tensor network scheme for modular S and T matrices"** to detect quantum phase transition numerically.

[Huang and Wei 2016]



- How to describe a quantum state?
  Tensor product states [F. Verstraete, Murg, & Cirac 2008]

- What is the "order parameter"?
  Modular matrices [Zhang,Grover, Turner, Oshikawa, & Vishwanath 2012]

- How to calculate the observable?
  Higher order tensor renormalization group [Xie, Chen, Qin, Zhu, Yang, & Xiang, 2012]

2D $\mathbb{Z}_N$ Topological order phase

* S & T from wave function overlaps (string/membranes as "symmetry twists"):
  ➔ **use real space renormalization to obtain fixed-point values**
  (as number of RG steps $n_{RG} \to \infty$);
  (note: symmetry twists are also coarse-grained)

* Ground-state degeneracy & modular matrices/invariants believed to be sufficient to characterize topological order

[Huang and Wei 2016]

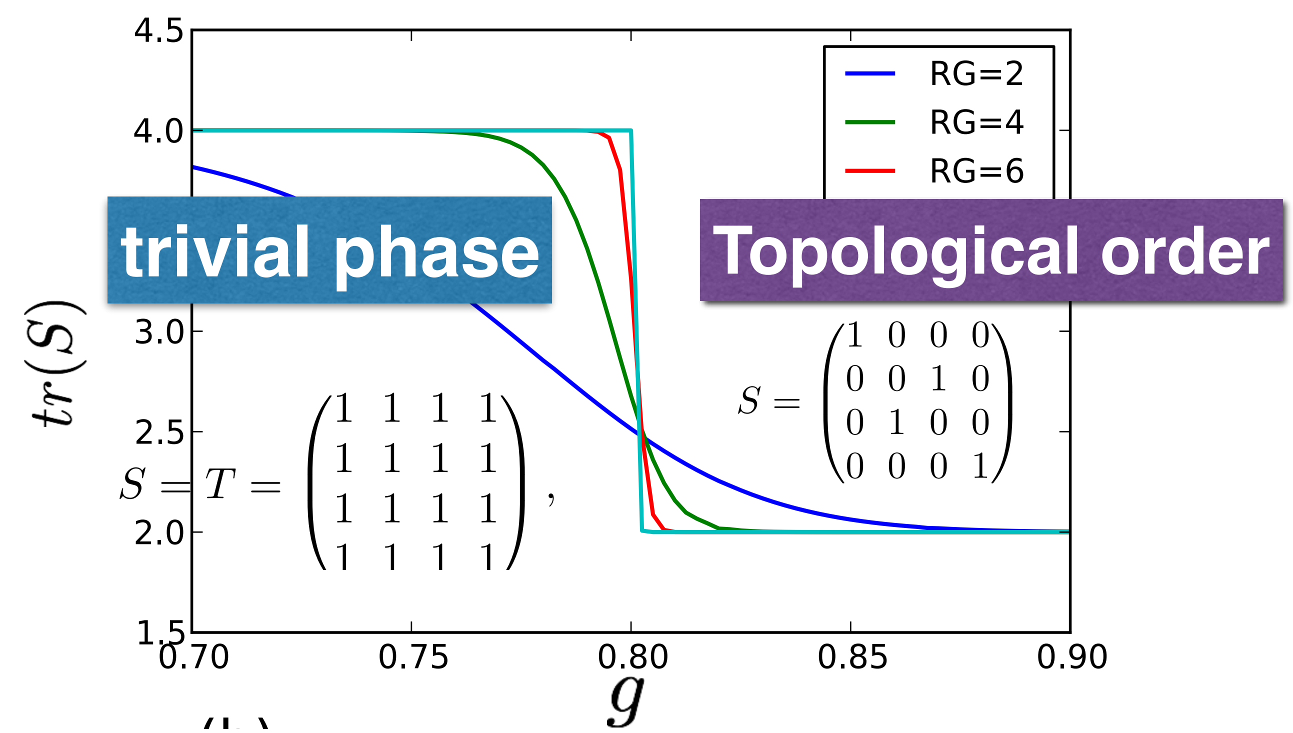
* $\mathbb{Z}_2$ topological order phase:

Wave function $|\Psi\rangle = \sum_c |\psi_c\rangle$

Deformed wave function

$$|\Psi(g)\rangle = Q(g) \otimes Q(g) \otimes Q(g) \otimes \ldots |\Psi\rangle$$

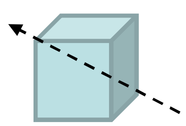
$$Q = |0\rangle\langle 0| + g|1\rangle\langle 1|$$



Topological invariant (Modular Matrices) in three dimension

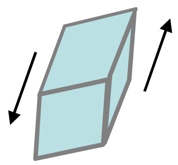
* SL (3,$\mathbb{Z}$) group : generated by a $\hat{s}$ and $\hat{t}$

$$\hat{s} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



cyclic shift of z,y,x axes

$$\hat{t} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



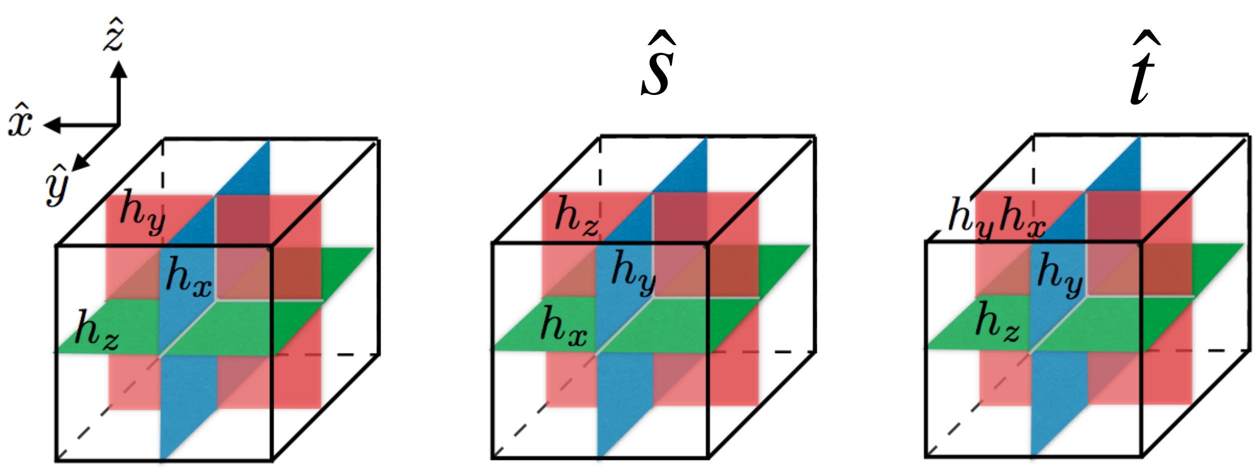
shear along y direction on surface $\perp$ x axis

* Modular matrices S and T are representations using degenerate ground states ➔ also give exchange/braiding statistics of anyonic excitations

$$S_{i,j} = \langle \Psi_i | \hat{s} | \Psi_j \rangle \quad T_{i,j} = \langle \Psi_i | \hat{t} | \Psi_j \rangle$$

* Ground states: membrane operators $\{\hat{h}_x, \hat{h}_y, \hat{h}_z\}$ acting on reference G.S.

$$|\Psi_j\rangle = \hat{h}_x \hat{h}_y \hat{h}_z |\Psi_0\rangle$$

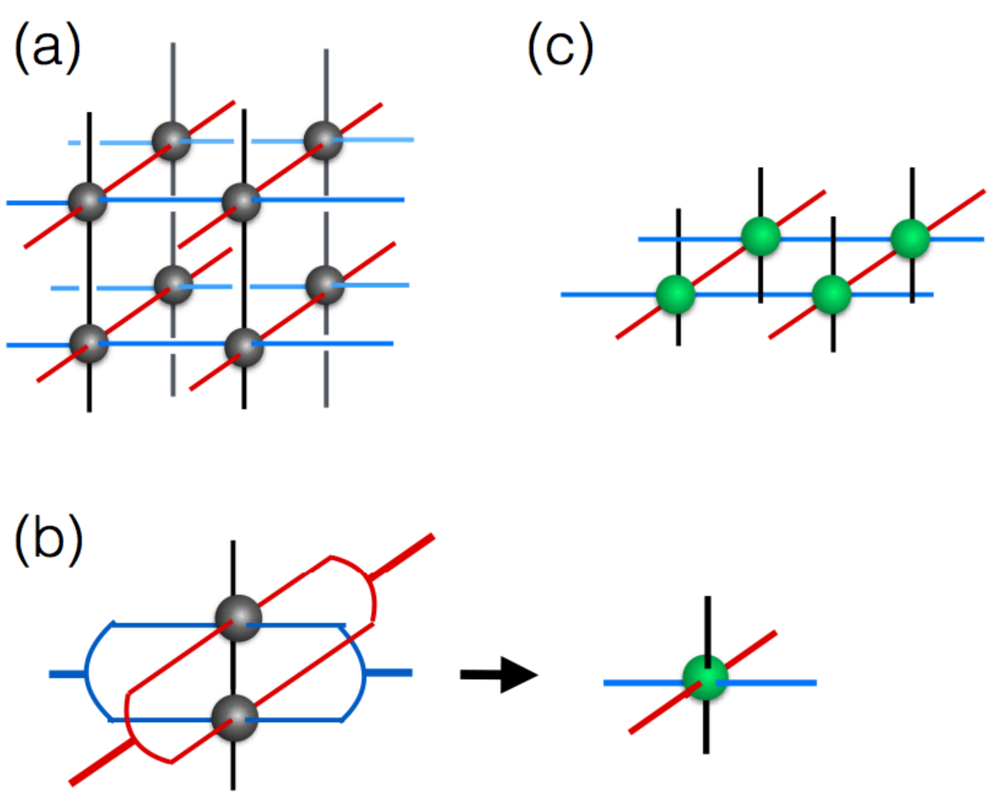


Use 3D HOTRG and 3D tnST scheme !!

Numerical method: 3D renormalization group

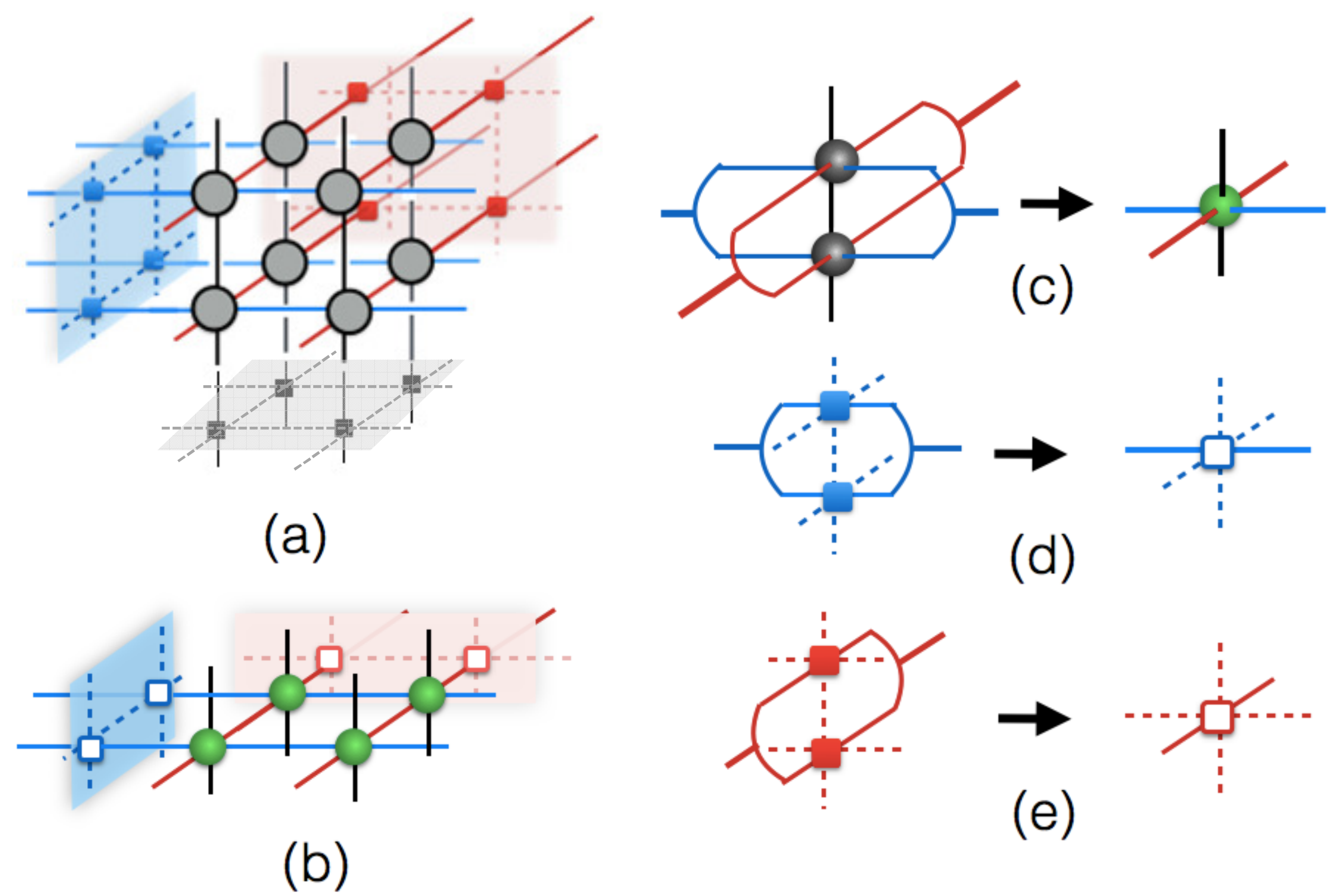
* 3D high order tensor renormalization group (HOTRG) [Xie,Chen, Qin, Zhu, Yang , Xiang,2012]

  → In the 3D calculation, the computational time scales with D^{11} and the memory scales with D^6.



3d HOTRG

* 3D tnST scheme :



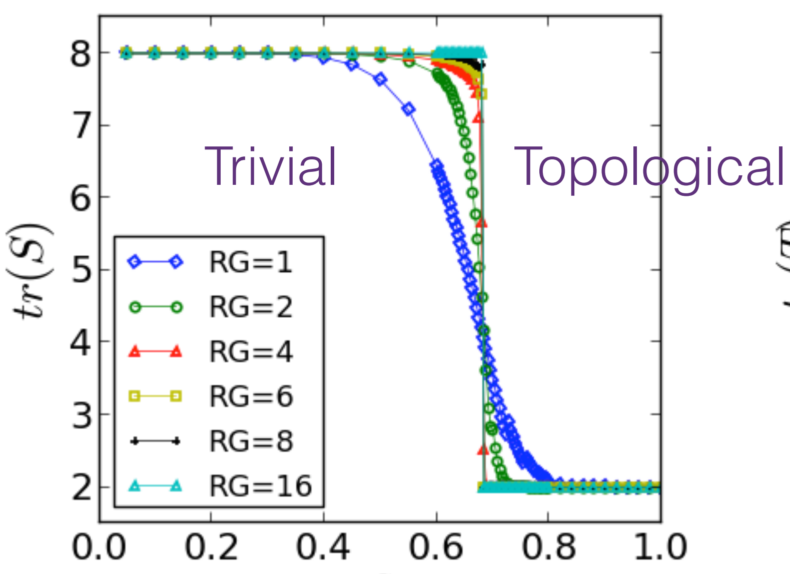
Numerical results:
3D $\mathbb{Z}_2$ topological order with deformation on cubic lattice

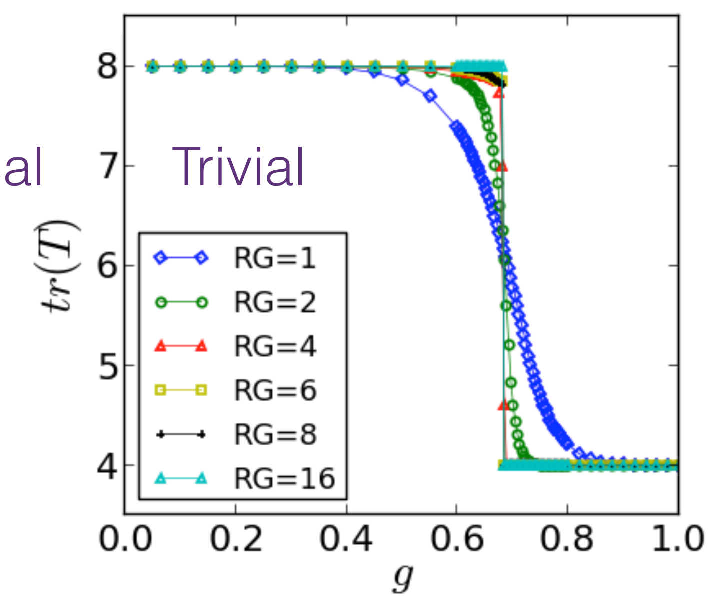
- Use **tr(S) and tr(T)** as "order parameters" [He,Moradi &Wen, PRB 14'] in 2D Z_2

- Deform the **3D toric-code ground state** by local operator $Q(g)$ on each spin

$$|\Psi(g)\rangle = Q(g)^{\otimes N}|\Psi_{TC}\rangle \qquad Q(g) = |0\rangle\langle 0| + g^2|1\rangle\langle 1|$$

(g=1: undeformed; g=0: product state)





$D_{cut} = 8$

Trivial phase

S,T =identity

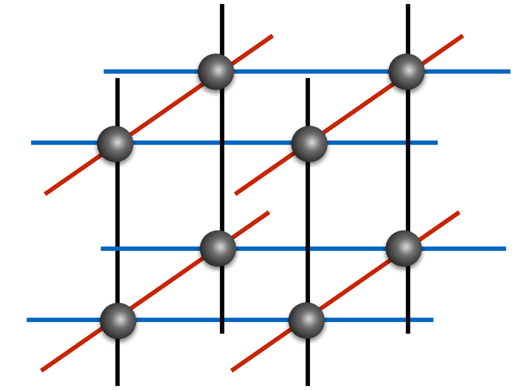
Topological order

$$S' = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix} \quad T' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

- Effective lattice size: $2^{3n_{RG}}$ (fixed point as RG steps $n_{RG} \to \infty$)

➔ transition at g≈0.68 from topological (e.g. g=1) to trivial phase (e.g. g=0)

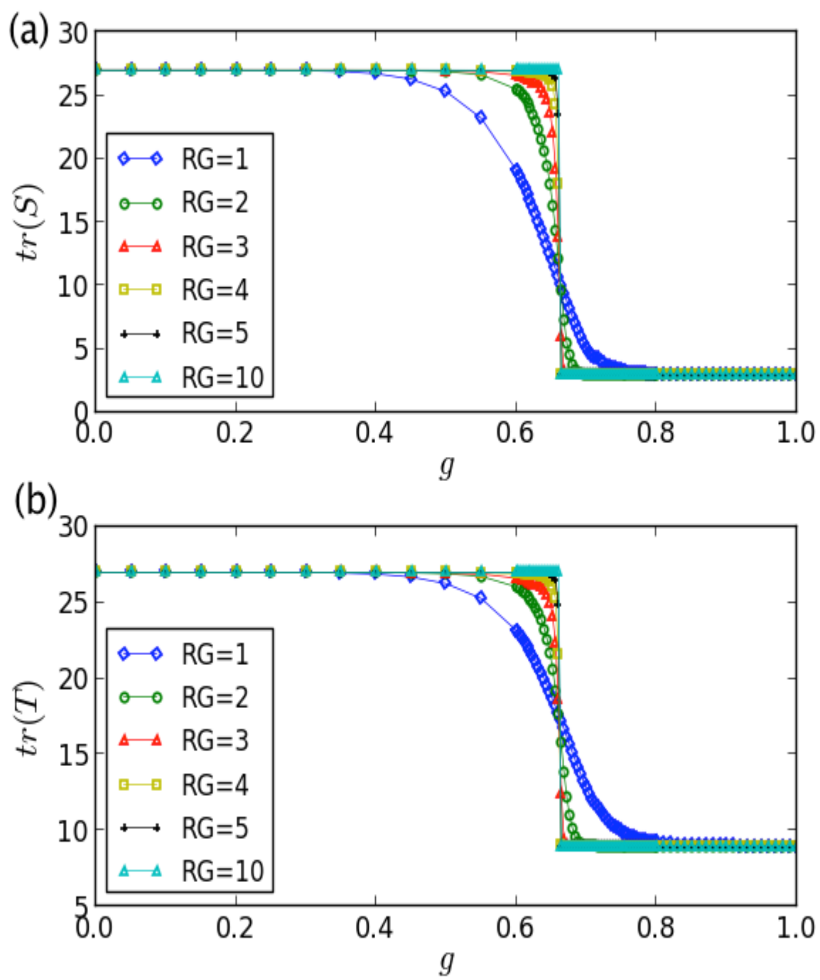
Numerical results:
Deforming $\mathbb{Z}_3$ and $\mathbb{Z}_4$ topological order



* Deform $\mathbb{Z}_3$:

$$Q(g)_{\mathbb{Z}_3} = |0\rangle\langle 0| + g^2|1\rangle\langle 1| + g^4|2\rangle\langle 2|$$

$$g_c \approx 0.66$$

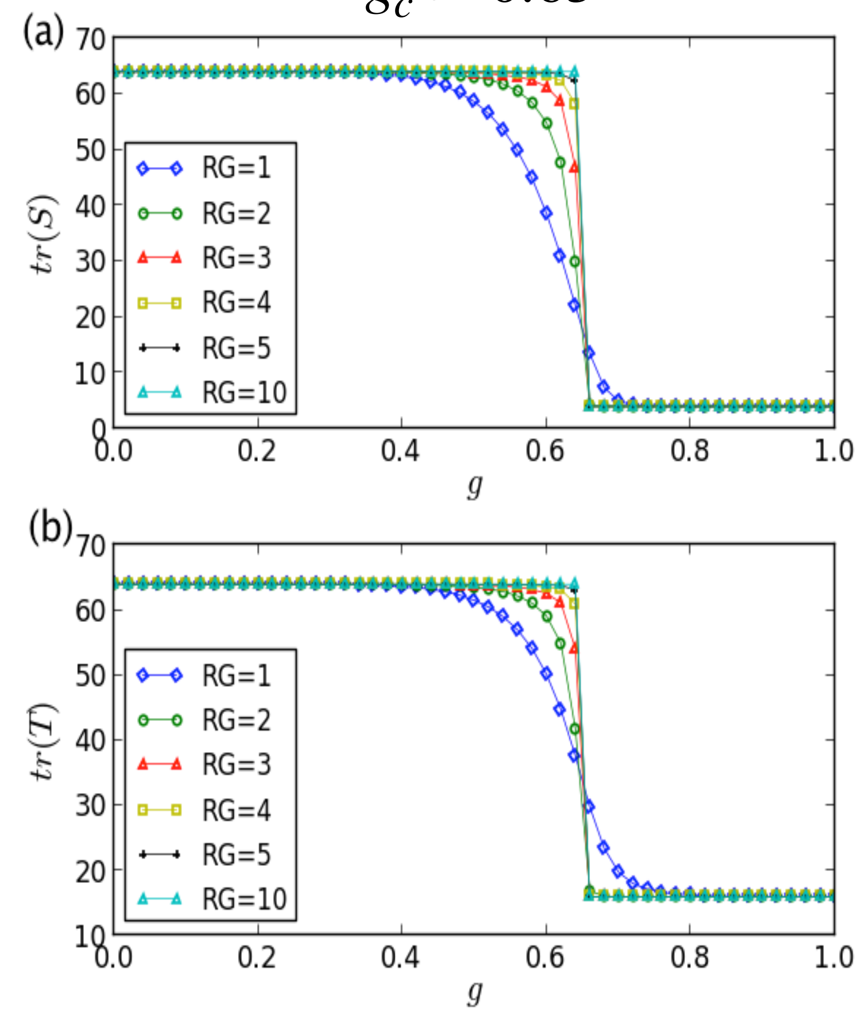


$$D_{cut} = 9$$

* Deform $\mathbb{Z}_4$:

$$Q(g)_{\mathbb{Z}_4} = |0\rangle\langle 0| + g^2|1\rangle\langle 1| + g^4|2\rangle\langle 2| + g^6|3\rangle\langle 3|$$

$$g_c \approx 0.65$$



$$D_{cut} = 8$$

3D $\mathbb{Z}_N$ topological order with deformation

* Transitions agree with mapping to 3D Ising/Potts models

* Under such deformation $Q = \sum_{i=0}^{N-1} q_i |i\rangle\langle i|$ and $q_i \geq 0$ ($q_0 = 1$ and $q_i = g^2$)

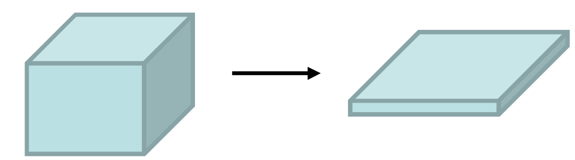
* $\langle \Psi_{GS}(g) | \Psi_{GS}(g) \rangle \iff \mathbb{Z}$ Potts partition function

$$g = \left(\frac{\sqrt{e^{\beta J} - 1}^2}{\sqrt{e^{\beta J} + N - 1}^2} \right)^{1/4}.$$

	Numerics	MC results	From mapping
N	g_c	βJ	$g_c(\beta J)$
2	0.68 $D_{cut} = 8$	0.443308	0.683378
3	0.66 $D_{cut} = 9$	0.5496	0.665594
4	0.65 $D_{cut} = 8$	0.6283	0.650802

Dimensional reduction: 3D →2D

* Compactify z-direction to small radius:
  (i) 3D → 2D (ii) SL(3,$\mathbb{Z}$) reduces to SL(2,$\mathbb{Z}$)



* 2D braiding is associated with SL(2,$\mathbb{Z}$) group, which is generated by

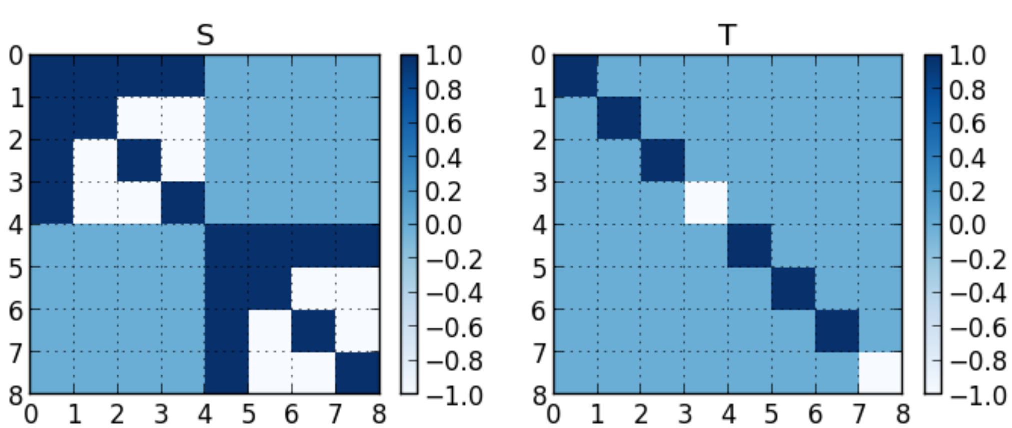
$$\hat{s}^{yx} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \hat{t}^{yx} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

➜ Reduction $C_G^{3D} = \bigoplus_{n=1}^{|G|} C_G^{2D}$ [Moradi & Wen 2015, Wang & Wen 2015]

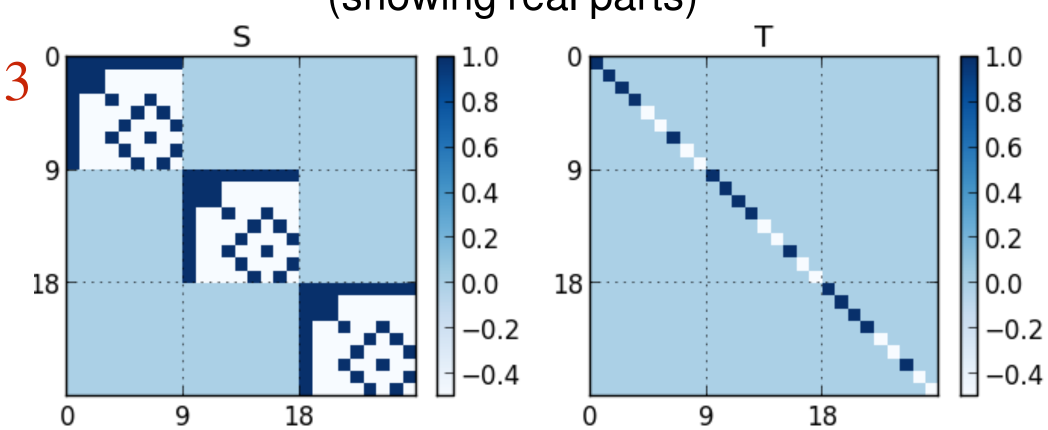
➜ We verify that 3D $\mathbb{Z}_N$ topological order is decomposed into copies of 2D $\mathbb{Z}_N$ topological order via block structure of S & T

(showing real parts)

$\mathbb{Z}_2$



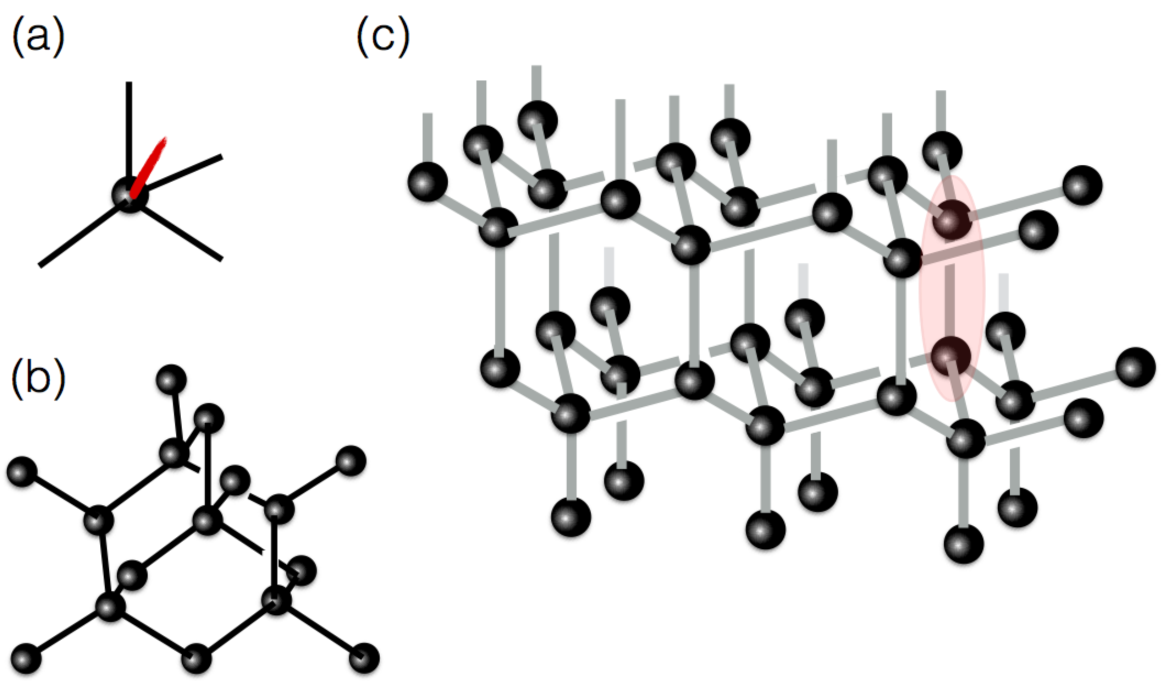
$\mathbb{Z}_3$

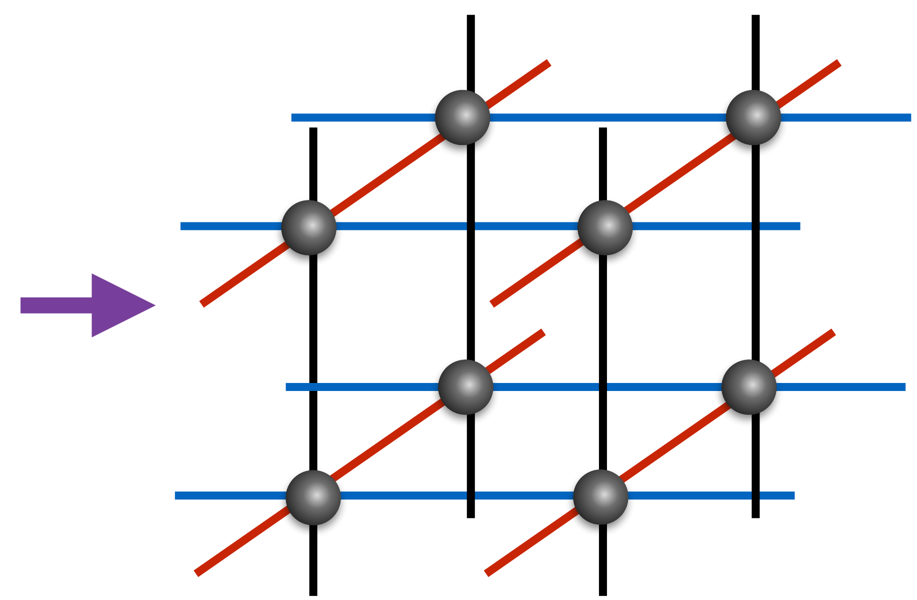


Other lattice structure

* Diamond lattice

➜ Combing two tensors to form a new tensor. The diamond lattice deforms into a cubic lattice.



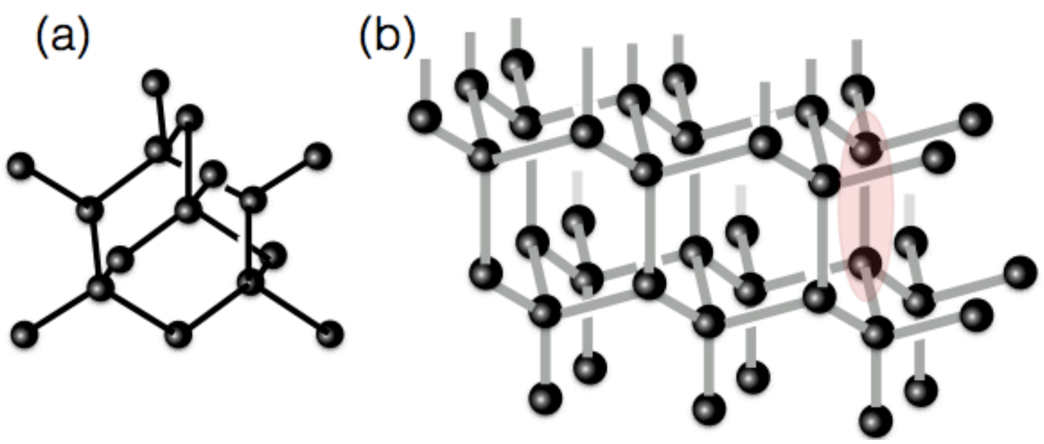


3d HOTRG

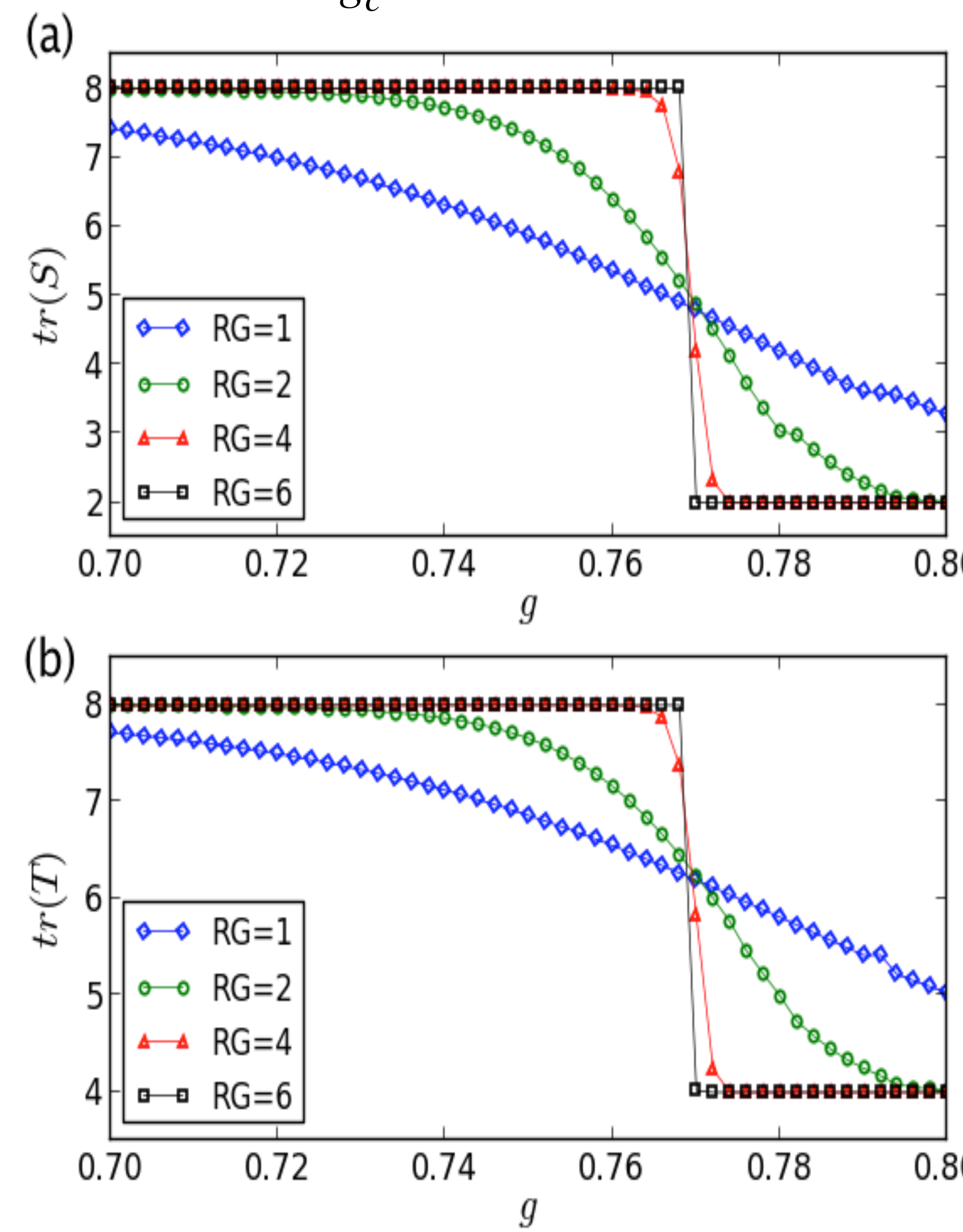
Deforming $\mathbb{Z}_2$ topological order in diamond lattice

* Deform $\mathbb{Z}_2$:

$$Q(g)_{\mathbb{Z}_2} = |0\rangle\langle 0| + g^2 |1\rangle\langle 1|$$



$g_c \approx 0.771$



Conclusion: part I

* **Main result:**

  tensor-network scheme for modular matrices (tnST) to diagnose 3D topological order

  $\rightarrow$ **successfully applied to transitions in 3D $\mathbb{Z}_N$ toric code under string tension**

* **Future:**

  1. Twisted "quantum double" models
  2. Fixed point wave function with deformation

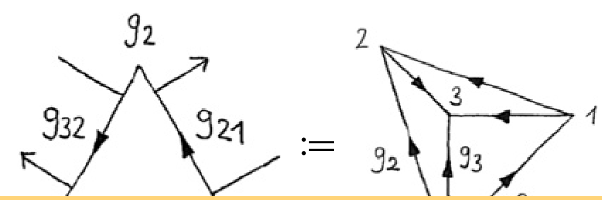
     -> exact MPO/ PEPO

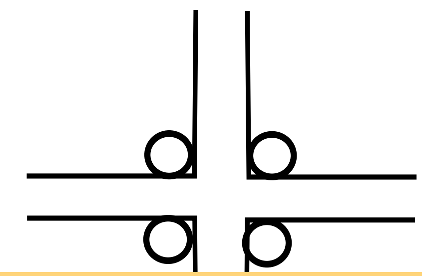
Twisted topological models

* 2d Twisted by 3-cocyle

$$TC : |\Psi\rangle = \sum_c |\psi_c\rangle \qquad DS : |\Psi\rangle = \sum_c (-1)^{\#\, loops} |\psi_c\rangle$$

[oliver, 2016]





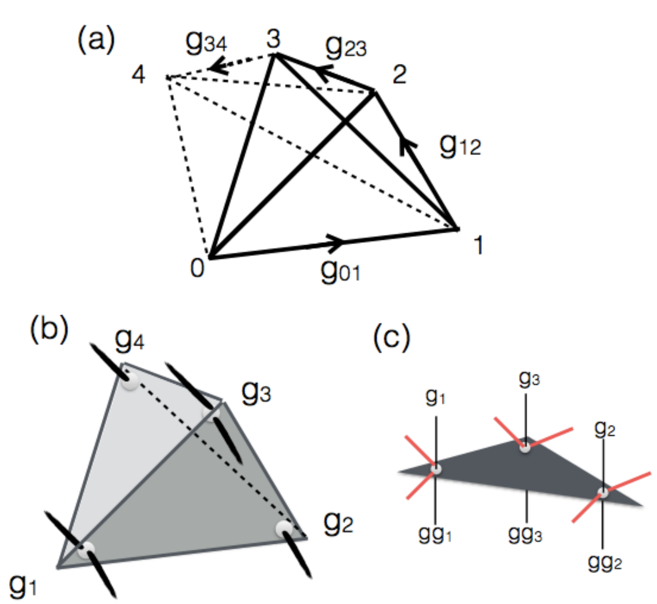
* 3d: Twisted by 4-cocyle

- The tensor representation of the basis vector
- The membrane operator

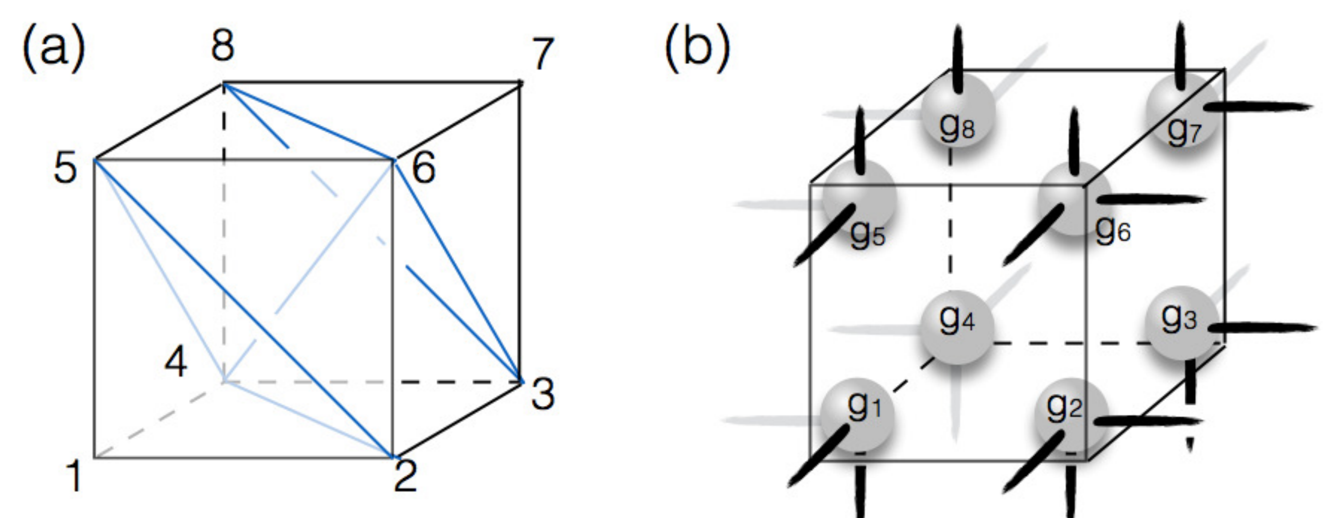
Need more efficient 3D tensor RG !!

ATRG, BTRG !!

* Tensor on cubic lattice: large physical degree and bond dimension



3d twisted TO



3d twisted TO

3D Twisted $Z_2 \times Z_2$ topological order

- From exact TO wave function
- GSD = 4^3 =64
- $H^4(Z_2 \times Z_2, U(1)) = (Z_2)^2$,
- The T matrix of w_{00}, from fixed point wave function

$$T = \bigoplus_{i=1}^{4} \begin{pmatrix}
1&&&&&&&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
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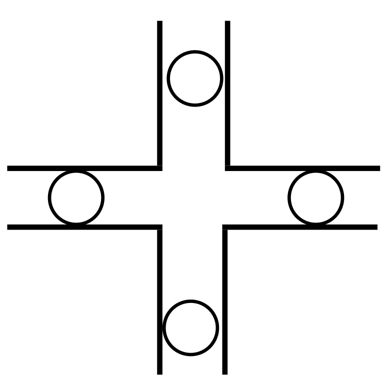
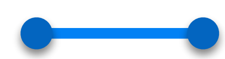
\end{pmatrix} \tag{40}$$



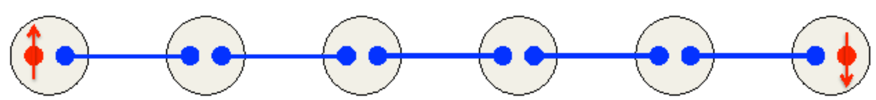
Order and disorder in AKLT antiferromagnets

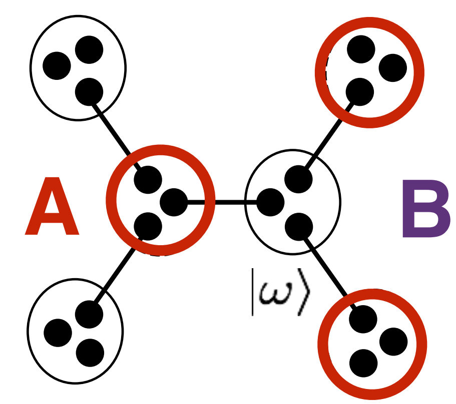
* valence-bond ground state

  simplest valence-bond of two spin-1/2 ➔ singlet state

  $|\omega\rangle = |01\rangle - |10\rangle$ 

* 1D and 2D structure [AKLT. 1987,1988]





* Affleck-Kennedy-Lieb-Tasaki (AKLT) state,

  state of spin 1, 3/2, or high (define on any lattice)

  ➔ unique ground state of two-body isotropic Hamiltonians

  $H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j)$ f(x) is a polynomial function

* AKLT states provides a **resource** for **universal quantum computation**

[Wei, Affleck and Raussendorf , 2011]

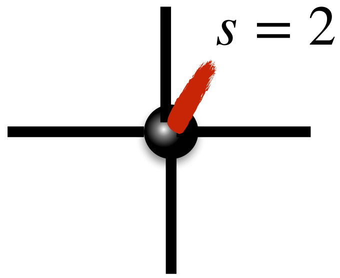
Previous work: Quantum Phase Transitions in Spin-2 AKLT Systems

* Proposal by Niggemann, Klu¨mper, and Zittartz, 2000

* Find Hamiltonian $H(a_1, a_2)$, which locally annihilates "deformed-AKLT" state

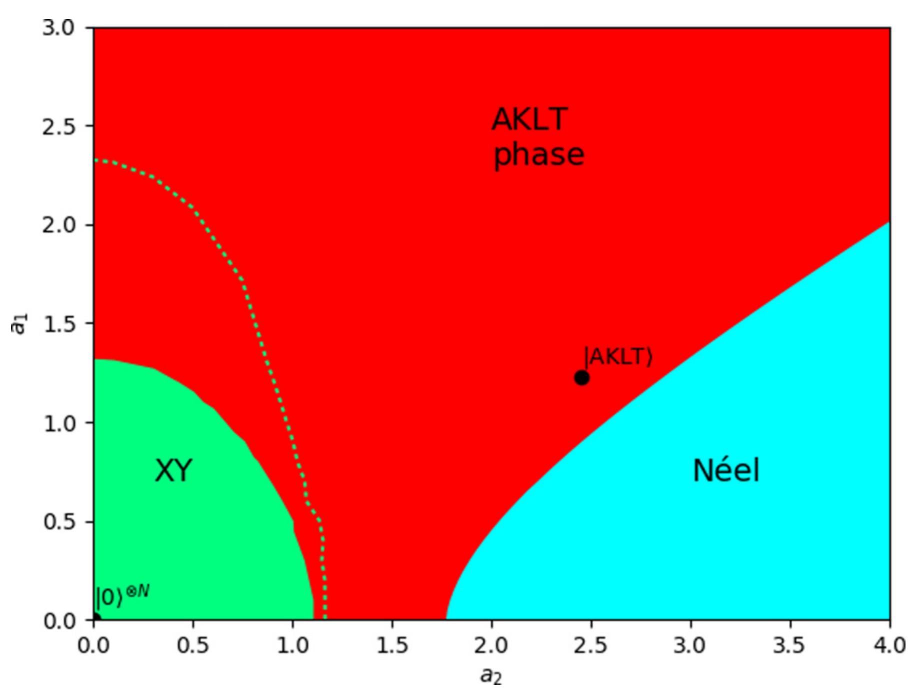
$$|\Psi(a_1, a_2)\rangle = Q(a_1, a_2)^{\otimes N} |\Psi_{AKLT}\rangle$$

$$Q(a_1, a_2) = |0\rangle\langle 0| + \sqrt{\frac{2}{3}} a_1(|1\rangle\langle 1| + |-1\rangle\langle -1|) + \sqrt{\frac{1}{6}} a_2(|2\rangle\langle 2| + |-2\rangle\langle -2|)$$

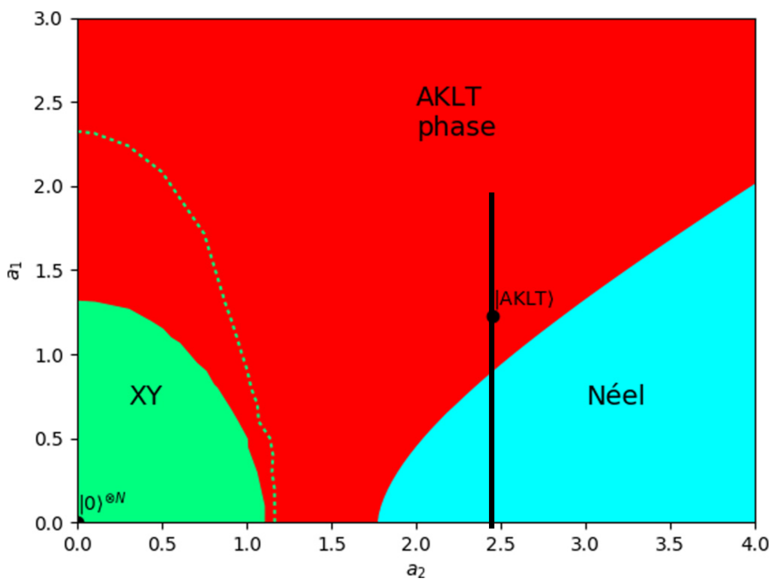


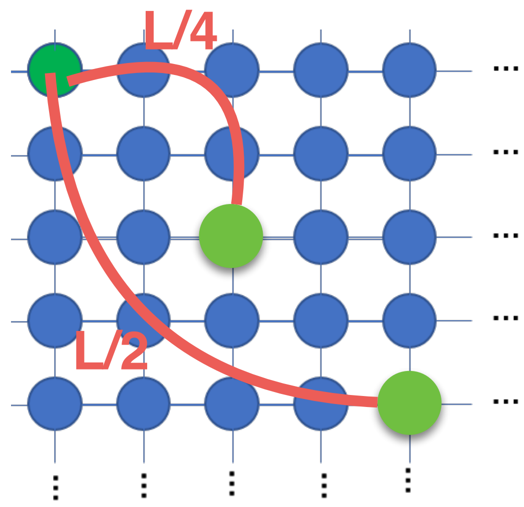
[Pomata ,Huang and Wei , 2018]

* correlation length (HOTRG)
* central charge (TNR)
* modular S & T matrices (tnST)



XY ↔ VBS: KT transition via a_1 [Huang ,Lu, and Chen ,in preparation]





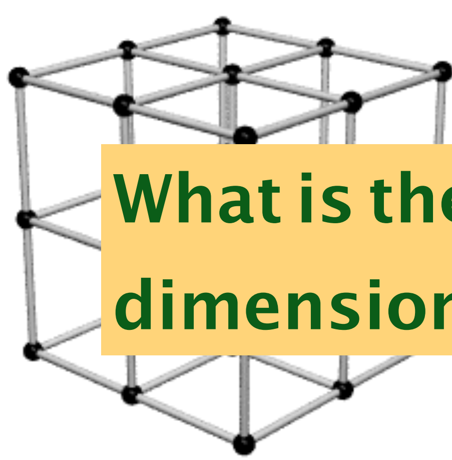
(1) Binder ratio U_2 $$U_2(a,L) = \frac{\langle m^4\rangle}{\langle m^2\rangle^2} = f((a-a_c)L^{1/\nu}).$$

[Morita, Kawashima,2018]

(2) correlation ratio $$R(a,L) \equiv \frac{C_{\max}(a,L)}{C_{\mathrm{halfmax}}(a,L)} = h_R(tL^{1/\nu}),$$

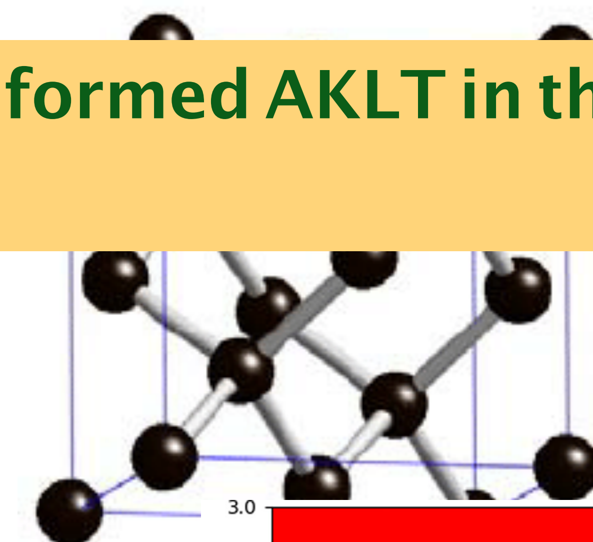
Order and disorder in AKLT antiferromagnets in three dimensions

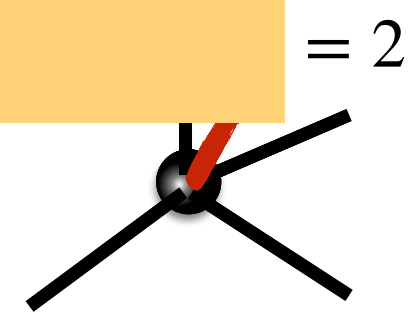
- ✱ AKLT state on cubic lattice (6 neighbors) : Neel state



- ✱ AKLT state on diamond lattice (4 neighbors) :disorder state

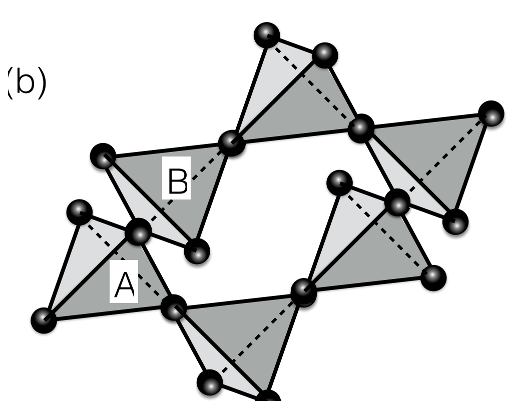
[Para meswaran,Sondhi, Arovas, 2009]

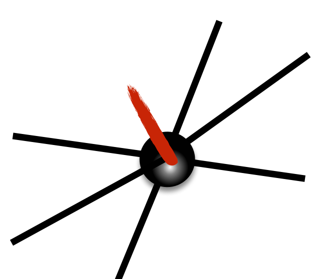


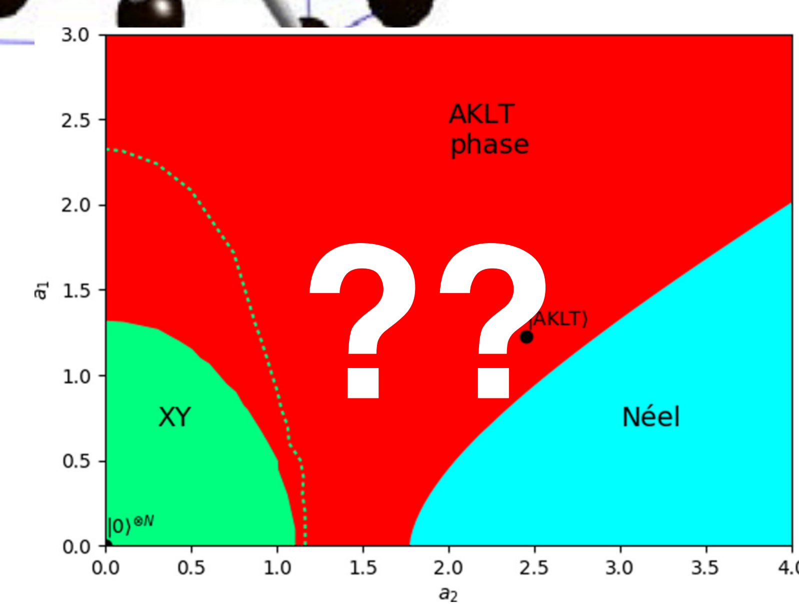


What is the phase diagram of the deformed AKLT in three dimensions?

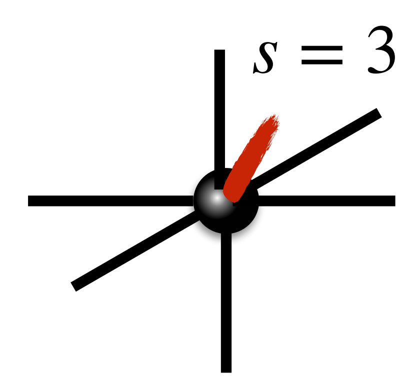
- ✱ AKLT state on pyrochlore (6 neighbors) : disorder state





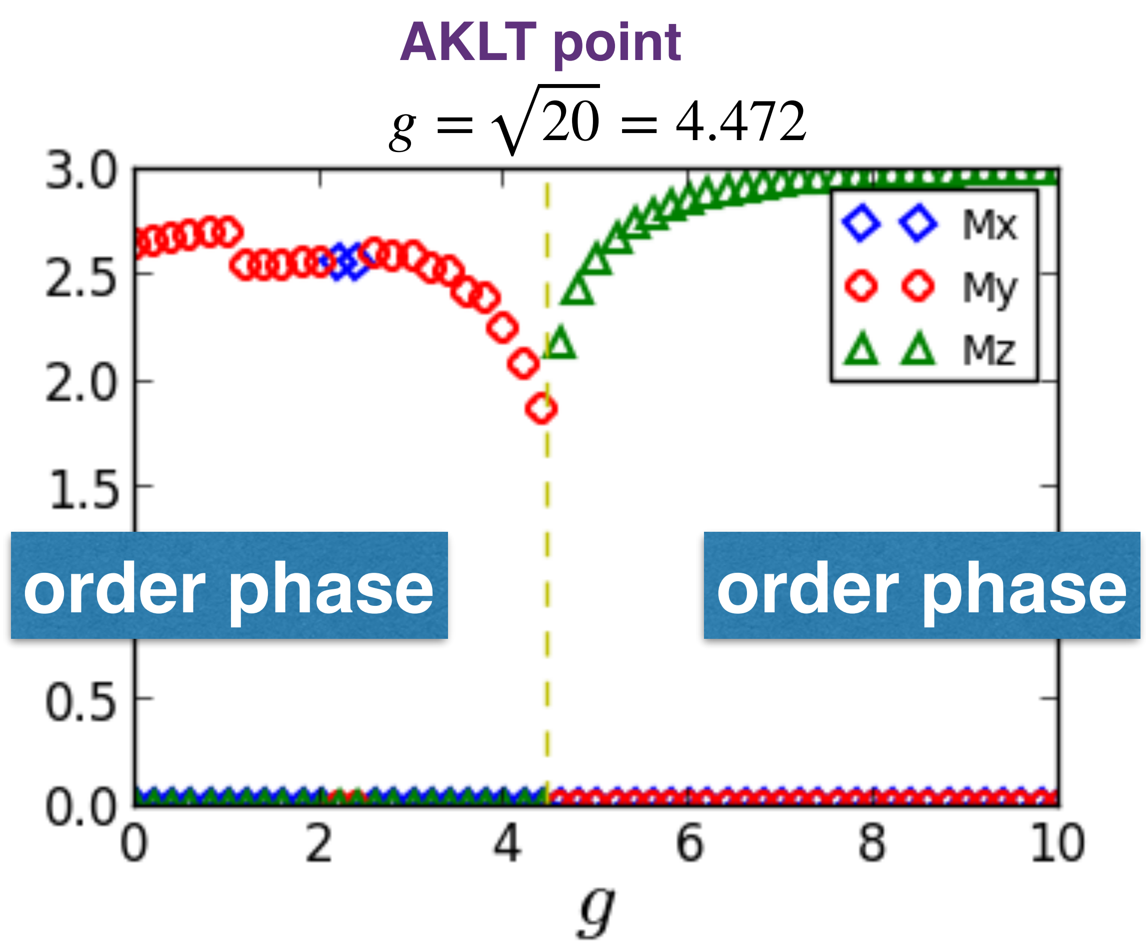


The spin-3 on the cubic lattice



* The deformed AKLT state $|\Psi(g)\rangle = Q(g)^{\otimes N}|\Psi_{AKLT}\rangle$

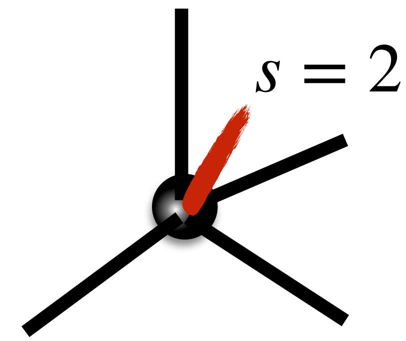
$$Q(g) = |0\rangle\langle 0| + \Big(|1\rangle\langle 1| + |-1\rangle\langle -1|\Big) + \Big(|2\rangle\langle 2| + |-2\rangle\langle -2|\Big) + \sqrt{\frac{1}{20}}g\Big(|2\rangle\langle 2| + |-2\rangle\langle -2|\Big)$$



The spin-2 on the diamond lattice

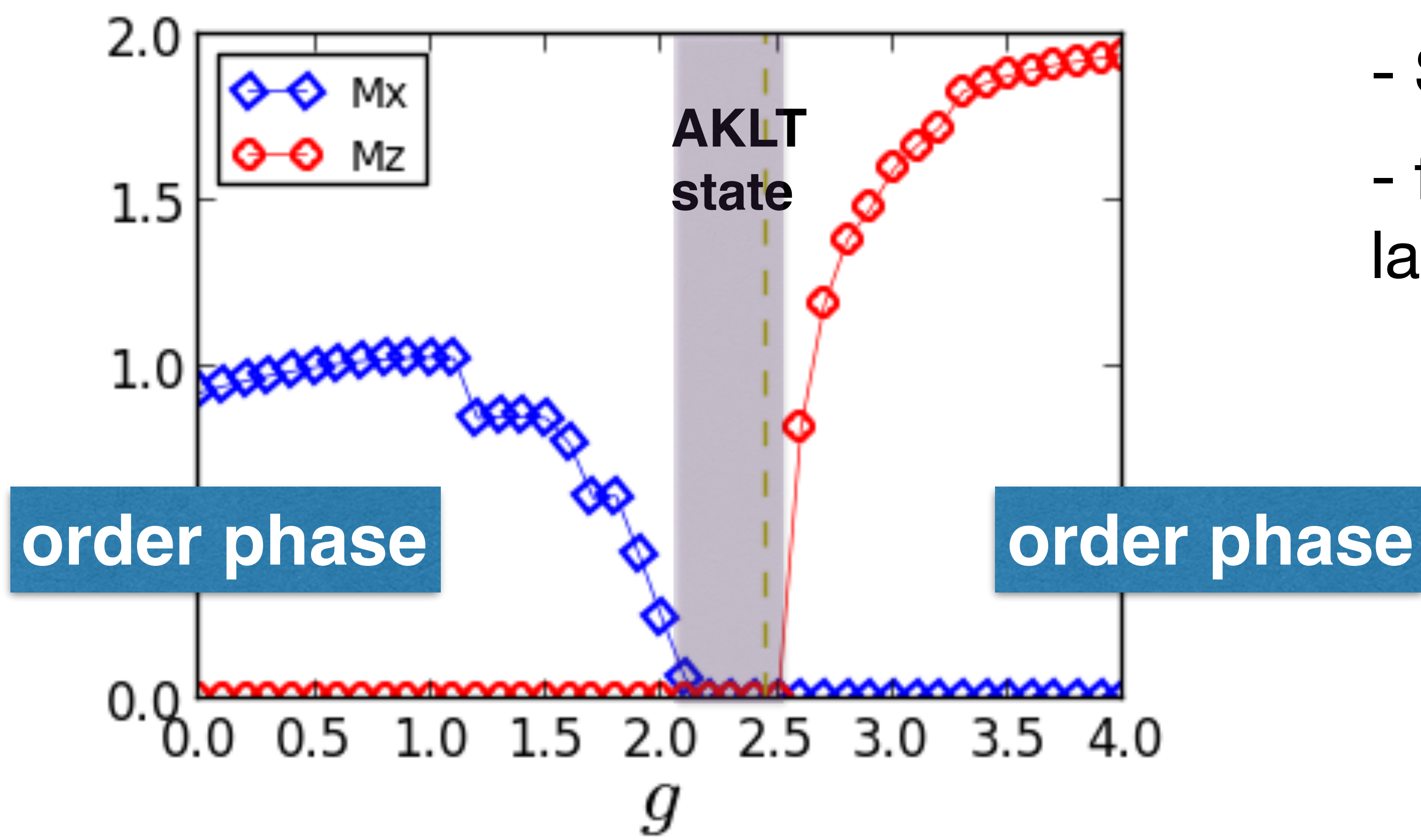
* The deformed AKLT state $|\Psi(g)\rangle = Q(g)^{\otimes N}|\Psi_{AKLT}\rangle$

$$Q(g) = |0\rangle\langle 0| + \Big(|1\rangle\langle 1| + |-1\rangle\langle -1|\Big) + \sqrt{\frac{1}{6}}g\Big(|2\rangle\langle 2| + |-2\rangle\langle -2|\Big)$$



AKLT point

$g = \sqrt{6} = 2.449$



Next step:

- S& T matrices
- finite size scaling (If we can large Dcut)….

Conclusion:

* **Main result:**

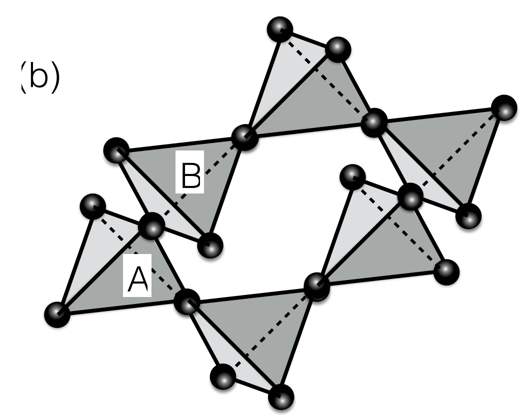
  1. tensor-network scheme for modular matrices (tnST) to diagnose 3D topological order

     → **successfully applied to transitions in 3D Zn toric code under string tension**

  2. study the one-parameter deformation of the AKLT state on the cubic lattice and the diamond lattice.

outlook

* find more efficiently RG scheme in 3D to fix phase boundary
* twisted topological order
* quantum state on <u>pyrochlore</u>

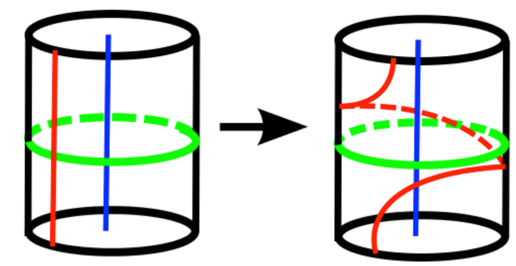


Thank you

T-matrix

- In toric code: $|\psi_{\alpha,\beta}\rangle = (\mathcal{Z}_1)^{\alpha}(\mathcal{Z}_2)^{\beta}|\psi_{0,0}\rangle$

$$T = \langle\psi_{\alpha',\beta'}|\hat{T}|\psi_{\alpha,\beta}\rangle$$



- Dehn twist

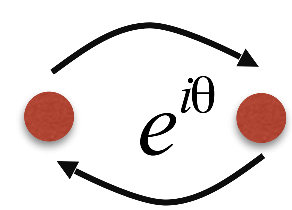
$$\boxed{|\psi_{\alpha,\beta}\rangle \to |\psi_{\alpha,\alpha+\beta}\rangle}$$

$$T = \begin{pmatrix} \langle\psi(\mathcal{I},\mathcal{I})|\psi(\mathcal{I},\mathcal{I})\rangle & \langle\psi(\mathcal{I},\mathcal{I})|\psi(\mathcal{I},\mathcal{Z})\rangle & \langle\psi(\mathcal{I},\mathcal{I})|\psi(\mathcal{Z},\mathcal{Z})\rangle & \langle\psi(\mathcal{I},\mathcal{I})|\psi(\mathcal{Z},\mathcal{I})\rangle \\ \langle\psi(\mathcal{I},\mathcal{Z})|\psi(\mathcal{I},\mathcal{I})\rangle & \langle\psi(\mathcal{I},\mathcal{Z})|\psi(\mathcal{I},\mathcal{Z})\rangle & \langle\psi(\mathcal{I},\mathcal{Z})|\psi(\mathcal{Z},\mathcal{Z})\rangle & \langle\psi(\mathcal{I},\mathcal{Z})|\psi(\mathcal{Z},\mathcal{I})\rangle \\ \langle\psi(\mathcal{Z},\mathcal{I})|\psi(\mathcal{I},\mathcal{I})\rangle & \langle\psi(\mathcal{Z},\mathcal{I})|\psi(\mathcal{I},\mathcal{Z})\rangle & \langle\psi(\mathcal{Z},\mathcal{I})|\psi(\mathcal{Z},\mathcal{Z})\rangle & \langle\psi(\mathcal{Z},\mathcal{I})|\psi(\mathcal{Z},\mathcal{I})\rangle \\ \langle\psi(\mathcal{Z},\mathcal{Z})|\psi(\mathcal{I},\mathcal{I})\rangle & \langle\psi(\mathcal{Z},\mathcal{Z})|\psi(\mathcal{I},\mathcal{Z})\rangle & \langle\psi(\mathcal{Z},\mathcal{Z})|\psi(\mathcal{Z},\mathcal{Z})\rangle & \langle\psi(\mathcal{Z},\mathcal{Z})|\psi(\mathcal{Z},\mathcal{I})\rangle \end{pmatrix}.$$

$$\boxed{T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

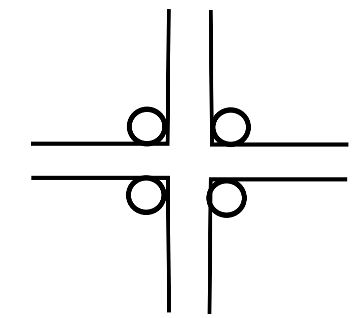
Use topological charge basis:

$$T = \begin{array}{c} \begin{array}{cccc} \mathbf{I} & \mathbf{e} & \mathbf{m} & \mathbf{em} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$



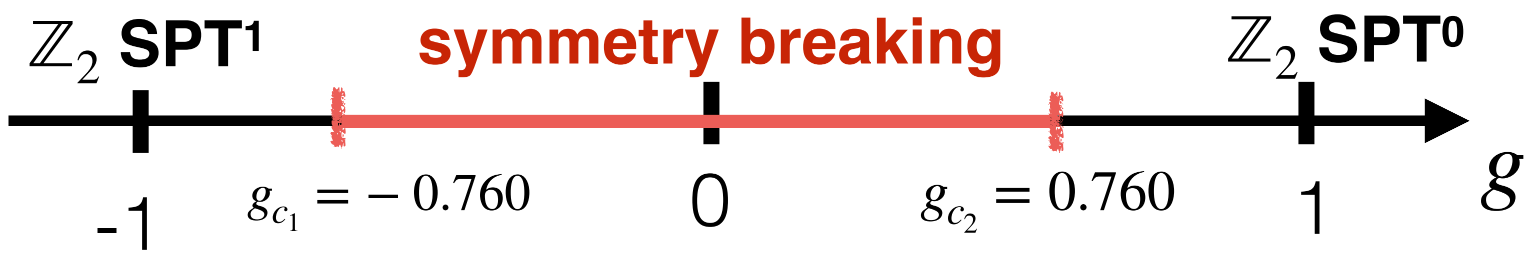
=> self statistics

2D $\mathbb{Z}_N$ (symmetry) topological order phase



2d SPT

* The $\mathbb{Z}_N$ SPT phase with deformation

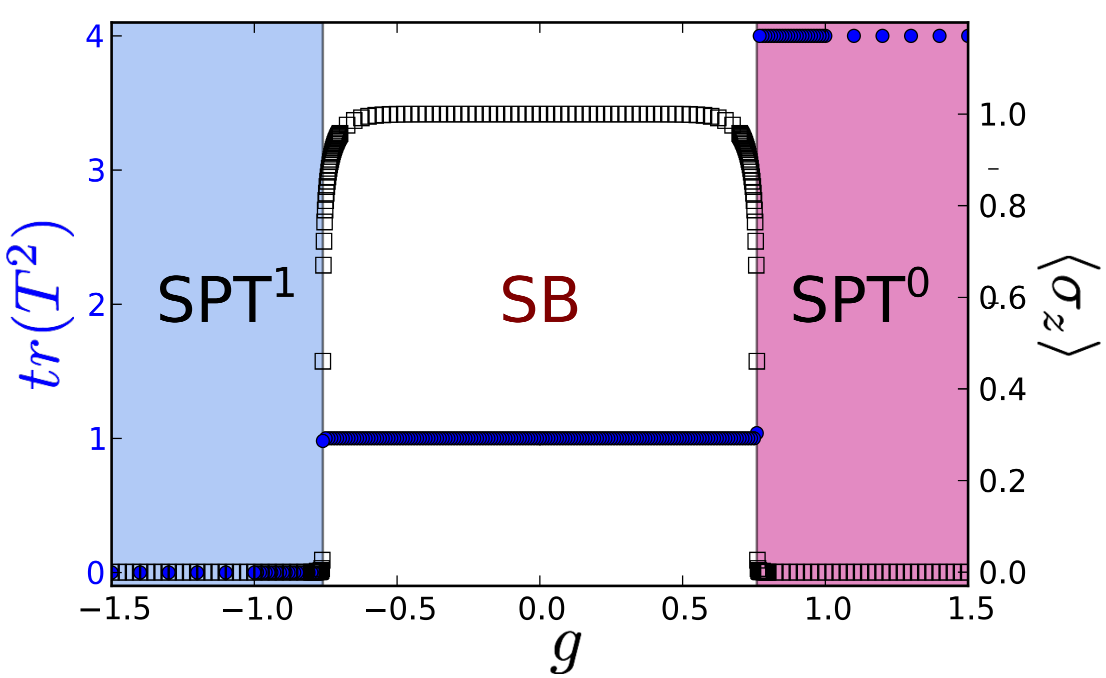


* Topological invariant

$$T^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^k & 0 \\ 0 & 0 & 0 & (-1)^k \end{pmatrix}$$

[Hung & Wen, 2014]

$$T^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



* The norm is equal to the partition function of **2D classical Ising model** on triangular lattice

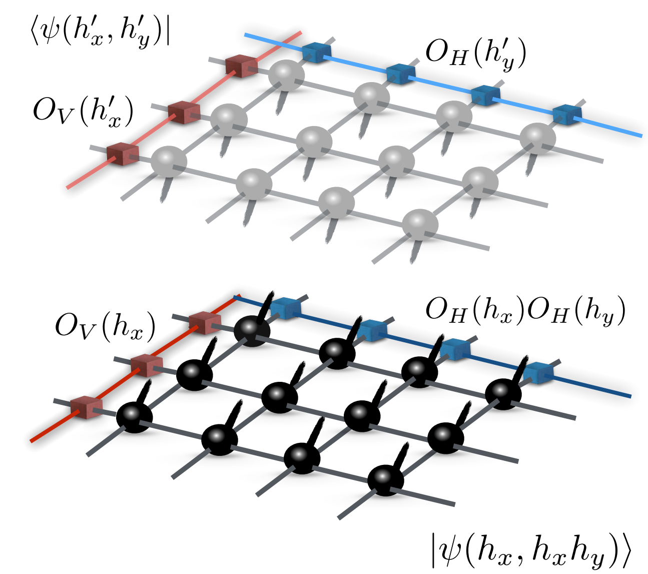
$$g_c = 3^{-0.25} = -0.759835$$

Tensor network scheme for modular S and T matrices (tnST)

* Creating the basis set $|\psi(h_x, h_y)\rangle$

  by inserting **string operator (TO)**

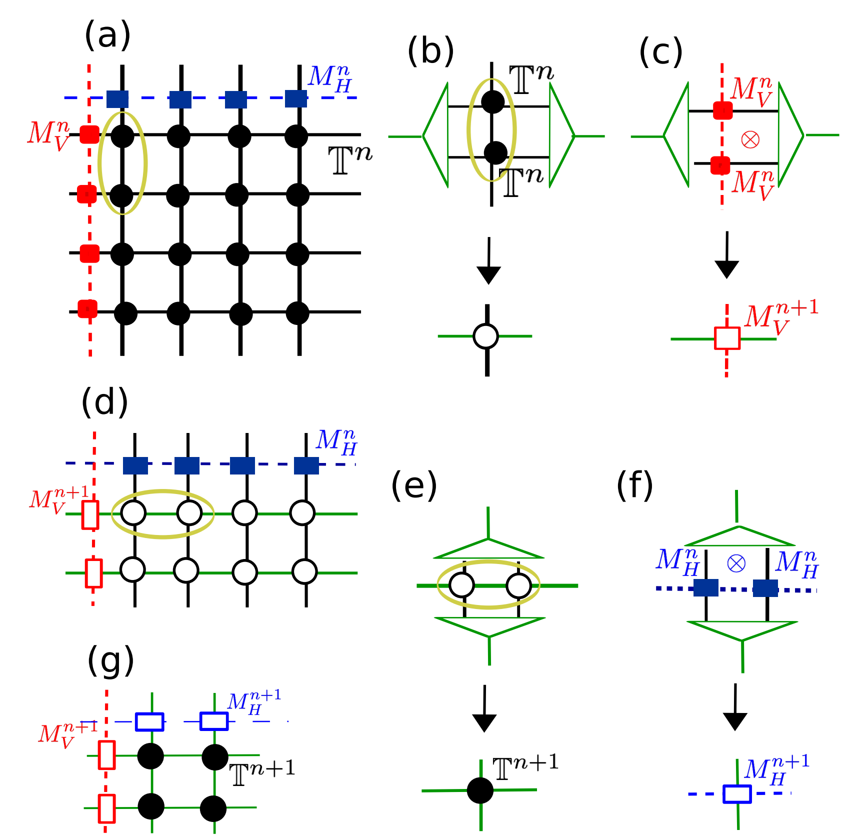
  symmetry twist (SPT)



* Simulating the rotation and the Dehn twist

$$\langle\psi(h'_x, h'_y)|\hat{t}|\psi(h_x, h_y)\rangle = \langle\psi(h'_x, h'_y)|\psi(h_x, h_x h_y)\rangle$$
$$\langle\psi(h'_x, h'_y)|\hat{s}|\psi(h_x, h_y)\rangle = \langle\psi(h'_x, h'_y)|\psi(h_y, h_x^{-1})\rangle.$$

* Creating the double tensor and double MPO's to determine the **wave function overlap**



[Huang and Wei 2016]