Understanding Kitaev-Related Models through Tensor Networks

2019.12.04 Naoki KAWASHIMA (ISSP)
Tensor Network (TN)

(1) Classical statistical mechanical models are TNs
--- corner transfer matrix (Baxter, Nishino, Okunishi, ...)

(2) TN is the "right" representation for renormalization group
--- "scale-invariant tensor", various techniques (TRG, TNR, loop-TNR, MERA, HOTRG, ...) (Gu, Levin, Wen, Vidal, Evenbly, Xiang, ...)

(3) TN is the "right" language for expressing topological/gauge structure
--- projective representation, symmetry-protected topological phases (Chen, Gu, Wen, Verstraete, Cirac, Schuch, Perez-Garcia, Oshikawa, ...)

(4) TN connects physics to informatics
--- application of/to data science, machine learning, automatic differentiation, lattice QCD, AdS/CFT-correspondence, etc.
Import from Stat. Mech. to TN

Example: Corner transfer matrix (CTM)

Nishino, Okunishi: JPSJ 65, 891 (1996)

Effect of the infinite environment is approximated by B and C, which are obtained by iteration/self-consistency.
TN-based real-space RG

Example: TNR (Evenbly-Vidal 2015)

Now, RSRG is not just an idea or a crude approximation, but a **systematic and precise** method.
Topological/gauge structure in TN

Example: Characterization of SPT phase

For the symmetry group G, an SPT phase is characterized by the 2nd cohomology group $H^2(G, U(1))$ of the projective representation of G.

Chen, Gu, Liu, Wen (2012)
Example: Image classification by TN

Zhao, Cichocki et al, arXiv:1606.05535

COIL100 2D image classification task
128 x 128 x 3 x 7200 bits

Decomposed the whole data into a tensor ring, and applied KNN classifier (K=1) to the image-identifying core ($Z_4$).

Ring decomposition shows better performance than open chain (TT).

We may use TN for identification of subtle characters.
Collaborators

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Kitaev Model


\[ H = \sum_{(ij)} \sum_{\mu=x,y,z} J_{ij}^{\mu} \sigma_i^{\mu} \sigma_j^{\mu} \]

\[ J_{ij}^{\mu} = \begin{cases} J & ((ij) \parallel \mu\text{-axis}) \\ 0 & \text{(otherwise)} \end{cases} \]

(1) **Symmetries** (rotation, translation, t-rev.)
(2) Flux-free
(3) Gapless (2D Ising Universality Class)
Loop Gas Projector (LGP)

\[ Q_{LG} = \sum_{\lambda_i, \mu_i, \nu_i = 0, 1} Q_{i}^{\lambda_i \mu_i \nu_i} \]

\[ \begin{align*}
0_0 & = 1 \\
1_0 & = \sigma_z \\
0_1 & = \sigma_x \\
1_1 & = \sigma_y
\end{align*} \]
LGP is projector to flux-free space

\[ P_{\text{LG}} \equiv \frac{1}{|\Gamma|} \sum_{\{\lambda_i\},\{\mu_i\},\{\nu_i\}} \prod_i Q_i^{\lambda_i \mu_i \nu_i} = \frac{1}{|\Gamma|} \sum_{G \in \Gamma} Q(G) \]

\[ Q(G) = \prod_i Q_i^{\lambda_i(G) \mu_i(G) \nu_i(G)} \]

\[ P_{\text{LG}}^2 = P_{\text{LG}} \quad W_p P_{\text{LG}} = P_{\text{LG}} W_p = P_{\text{LG}} \]

Loop Gas State \[ |\psi_0\rangle \equiv |\text{LGS}\rangle \equiv P_{\text{LG}} |(111)\rangle \]

LGS is flux-free and non-magnetic
Loop Gas State

\[ |\text{LGS}\rangle = \sum_{G: \text{loop config.}} Q(G) |(111)\rangle \]

\[ |(111)\rangle \ldots \text{fully-polarized state in (111) direction} \]

\[ \langle \text{LGS}|\text{LGS}\rangle = Z_{\text{LG}} \left( \zeta = \frac{1}{\sqrt{3}} \right) = Z_{\text{Triangular Ising}}(T = T_c) \]

Exactly at the critical point.
Classical Loop Gas


\[ Z_{LG}(n, \zeta) \equiv \sum_{G: \text{loop config.}} n^{N_{\text{loop}}(G)} \zeta^{|G|} \]

\[ \zeta_c(n) = \frac{1}{\sqrt{2 + \sqrt{2 - n}}} \]

\[ \zeta_c(1) = \frac{1}{\sqrt{3}} \]

Gapped \hspace{1cm} Critical

\[ \zeta_c \hspace{1cm} \zeta \]

LGS is gapless and belongs to the 2D Ising universality class (the same as the KHM ground st.)
Better variational function $\psi_1$

$$\psi_1 \equiv R_{DG}(\phi)|LGS\rangle$$

$R_{DG}(\phi)$ is analogous to $P_{LG}$ with loops replaced by dimers. (tan $\phi$ is the fugacity of the dimers)

2D Ising universality class for any $\varphi$
Series of Ansatzes

\begin{align*}
|\psi_0\rangle & \equiv P_{LG}|(111)\rangle = |LGS\rangle \\
|\psi_1\rangle & \equiv P_{LG} R_{DG}(\phi_1) |(111)\rangle \\
|\psi_2\rangle & \equiv P_{LG} R_{DG}(\phi_1) R_{DG}(\phi_2) |(111)\rangle
\end{align*}

<table>
<thead>
<tr>
<th># of prmtrs.</th>
<th>(\psi_0)=LGS</th>
<th>(\psi_1)</th>
<th>(\psi_2)</th>
<th>KHM gr. st.</th>
</tr>
</thead>
<tbody>
<tr>
<td># of prmtrs.</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>E/J</td>
<td>-0.16349</td>
<td>-0.19643</td>
<td>-0.19681</td>
<td>-0.19682</td>
</tr>
<tr>
<td>(\Delta E/E)</td>
<td>0.17</td>
<td>0.02</td>
<td><strong>0.00007</strong></td>
<td></td>
</tr>
</tbody>
</table>

Best accuracy by numerical calculation is achieved with only two tunable parameters
Kitaev Model on Star Lattice

\[
\hat{\mathcal{H}} = -\frac{J}{4} \sum_{\langle i,j \rangle \in \gamma} \hat{\sigma}_i^\gamma \hat{\sigma}_j^\gamma - \frac{J'}{4} \sum_{\langle i,j \rangle \in \gamma'} \hat{\sigma}_i^{\gamma'} \hat{\sigma}_j^{\gamma'}
\]

The ground state is CSL


On the torus,
Abelian CSL ... 4-fold degenerate
non-Abelian CSL ... 3-fold degenerate
Effect of Global-Flux Operator

Global flux op. is equivalent to the global gauge twist.
Minimally Entangled States (MES)

Zhang, Grover, Turner, Oshikawa, and Vishwanath, PRB85, 235151 (2012)

For $|\psi\rangle = |LG\rangle$ or $|SG\rangle$,

$$|I\rangle = |\psi\rangle + \hat{\Phi}_y |\psi\rangle \quad \text{(trivial)}$$

$$|m\rangle = |\psi\rangle - \hat{\Phi}_y |\psi\rangle \quad \text{(vortex)}$$
Topological Entanglement Entropy

\[ S = \alpha L_y - \gamma_i \]

TEE depends on the top. excitation (Ising anyon)

TEE does not depend on the top. excitation (the same as toric code)

Conformal Information of Chiral Edge Modes

$L_y = 6$ at $\phi = 0.25\pi$ (in non-Abelian phase)

Clear evidence for the chiral edge mode characterized by the Ising universality class

Entanglement Spectrum: Li and Haldane, PRL 101, 010504 (2008)

S=1 Kitaev Model


\[ W_p \equiv e^{i\pi(S^y_1 + S^z_2 + S^x_3 + S^y_4 + S^z_5 + S^x_6)} \quad \text{(any S)} \quad [H, W_p] = 0 \]

The ground state is non-magnetic.

Exact diagonalization N<=24

Gapless?

2-step structure analogous to S=1/2
Loop-gas projector for $S=1$

\[ P_{LG} = \]

$U_z \equiv \prod_{i \in \Gamma_z} U_i^z$

the global flux operator

$W_z \equiv \prod_{i \in \Gamma_z} U_i^z$

$\sigma$ replaced by $\pi$-rotations

\[ U^\gamma = e^{i\pi S^\gamma} \]

A LG projected state is always an eigenstate of the global flux operator.
S=1 loop-gas state

Fugacity of the loop gas is

\[ \langle (111) | U^\gamma | (111) \rangle = \begin{cases} 
\frac{1}{\sqrt{3}} & (S = \frac{1}{2}) \\
\frac{1}{3} & (S = 1)
\end{cases} \]

\[ \zeta(S=1) = 1/3 < \zeta_c = 1/\sqrt{3} = \zeta(S=1/2) \]

S=1 loop-gas state is gapped.
All minimally entangled states have the same topological entropy.

--> Abelian spin liquid

(Zhang et al: PRB85 235151 (2012))
Summary

New framework for describing Kitaev spin liquids

(1) $S=\frac{1}{2}$ Kitaev honeycomb model (KHM)
   - both critical and gapped cases are expressed by TN
   - KHM-LGS relation (analogous to AFH-AKLT relation)

(2) $S=\frac{1}{2}$ star-lattice Kitaev model
   - both Abelian and non-Abelian gapped chiral SL
     are expressed as TN (CF: the no-go theorem by Dubail-Read (2015))
     - Conformal tower of the Ising CFT

(3) $S=1$ Kitaev honeycomb model
   - gapped Abelian SL
END