# Application of Tensor Network States to Lattice Field Theories

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PERIMETER INSTITUTE

TNSAA 2019 – 2020

National Chengchi University

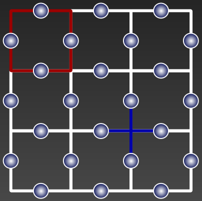
December 2019

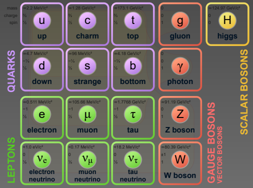
# Outline

# Motivation

## Gauge field theories

- (Gauge) field theories are central for many aspects in physics
  - Condensed matter physics
    - Toric Code
    - Topological order
    - Fractional quantum Hall states
  - Standard model of particle physics
- Non-perturbative regime: analytical access hard





# Motivation

## Lattice Field Theory

- Conventional approach: discretize space-time on a lattice
- After Wick rotation the path integral can be evaluated numerically on the lattice

$$\langle O[\phi] \rangle = \frac{\int \mathcal{D}\phi \, O[\phi] \exp(iS_M[\phi])}{\int \mathcal{D}\phi \exp(iS_M[\phi])}$$

$$\xrightarrow[t \to -i\tau]{} \int \mathcal{D}\phi \, O[\phi] \boxed{\frac{\exp(-S_E[\phi])}{\int \mathcal{D}\phi \exp(-S_E[\phi])}}$$

$p(x)$

- $p(x)$ is a probability if $\exp(-S_E[\phi])$ is real and nonnegative

$\Rightarrow$ We can use Monte Carlo methods to compute $\langle O[\phi] \rangle$

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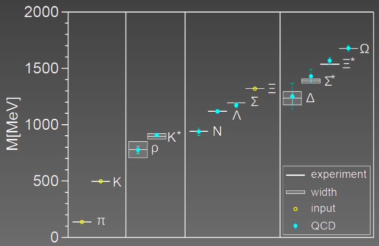
## Lattice Field Theory

- Highly successful for static properties such as mass spectra

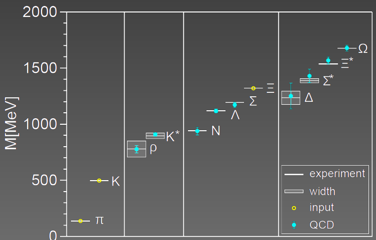


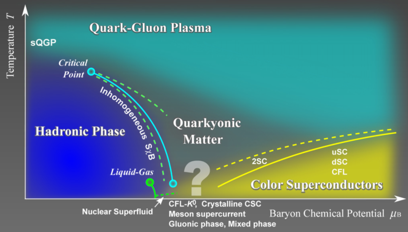

S. Dürr et. al., Science 322 1224 (2008)

# Motivation

## Lattice Field Theory

- Highly successful for static properties such as mass spectra
- No real-time dynamics
- Sign problem/complex action problem even for certain static problems




S. Dürr et. al., Science 322 1224 (2008)
K. Fukushima and T. Hatsuda, Rep. Prog. Phys. 74, 014001 (2011)

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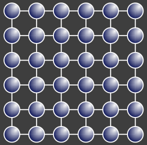
Tensor Networks as a numerical tool to solve Lattice Field Theories

## Computing partition functions

- Write the partition function as a Tensor Network

$Z =$ 

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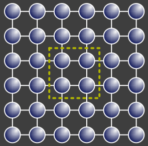
$Z =$ 

- Contract the resulting network (approximately)

$\Rightarrow$ TRG and TNR approaches

M. Levin, C. P. Nave, Phys. Rev. Lett. 99, 120601 (2007)
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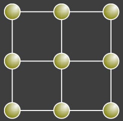
$Z =$ 

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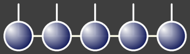
- TNS as a variational class

$|\psi\rangle =$ 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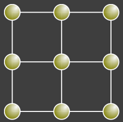
$Z =$ 

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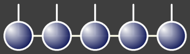
$|\psi\rangle =$ 

- Numerical algorithms for
  - ✓ Ground states
  - ✓ Low-lying excitations
  - ✓ Thermal states
  - ○ Time evolution


G. Vidal, Phys. Rev. Lett. 93, 040502 (2004)
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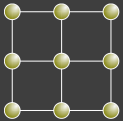
$Z =$ 

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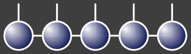
$|\psi\rangle =$ 

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**In both approaches there is no sign problem!**


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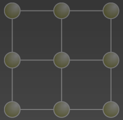
$Z =$ 

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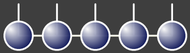
$|\psi\rangle =$ 

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# Motivation

## General strategy

- Hamiltonian formulation
  - Choose a Hilbert space with an appropriate basis
- Continuous (gauge) symmetries lead to infinite dimensional Hilbert spaces
  - Truncate the basis
  - Quantum link models
  - Integrate the gauge degrees of freedom out

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## General strategy

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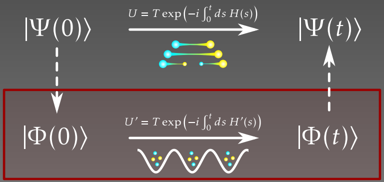
$\Rightarrow$ Common ingredients for quantum simulation



# The Schwinger model/QED in 1+1 dimensions

## Continuum formulation

- Euclidian time Lagrangian of the model

$$\mathcal{L} = \underbrace{\bar{\psi}\gamma_\mu D^\mu \psi}_{\text{kinetic energy + coupling to the gauge field}} + \underbrace{m\bar{\psi}\psi}_{\text{mass term}} + \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{dynamics of gauge field}}$$

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \; D_\mu = \partial_\mu + igA_\mu, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

J. Schwinger, Phys. Rev. 128 2425 (1962)

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- Simplest nontrivial gauge theory with matter
- Exactly solvable in the massless case
- Many similarities with QCD
  - Confinement
  - Chiral symmetry breaking

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$$\mathcal{L} = \bar{\psi}\gamma_\mu D^\mu \psi + m\bar{\psi}\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\frac{g}{2}\frac{\theta}{2\pi}\varepsilon_{\mu\nu}F_{\mu\nu}$$

kinetic energy + coupling to the gauge field | mass term | dynamics of gauge field | topological $\theta$-term

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \; D_\mu = \partial_\mu + igA_\mu, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \; \theta \in [0, 2\pi)$$

- Simplest nontrivial gauge theory with matter
- Exactly solvable in the massless case
- Many similarities with QCD
  - Confinement
  - Chiral symmetry breaking
  - Theta vacua, strong CP problem

$\Rightarrow$ Action is no longer real if $\theta \neq 0$, sign problem

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# The Schwinger model

## Lattice Hamiltonian formulation

- Kogut-Susskind staggered fermions in temporal gauge $A^0 = 0$

$$H = -\frac{i}{2a}\sum_{n=1}^{N-1}\left(\phi_n^\dagger e^{i\alpha_n}\phi_{n+1} - \text{h.c.}\right) + \sum_{n=1}^{N}(-1)^n m\,\phi_n^\dagger\phi_n + \frac{ag^2}{2}\sum_{n=1}^{N-1}\left(L_n + \frac{\theta}{2\pi}\right)^2$$

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kinetic part + coupling to gauge field

staggered mass term

electric energy

$\phi_n$: single-component fermionic field

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kinetic part + coupling to gauge field — staggered mass term — electric energy

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- Gauss Law

$$L_n - L_{n-1} = Q_n = \phi_n^\dagger \phi_n - \frac{1}{2}\left(1 - (-1)^n\right)$$

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$l_0$

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$l_0 + Q_1 = l_1$

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## Numerical simulation with Matrix Product States

- Map fermions to spins with a Jordan-Wigner transformation
  $\phi_n = \prod_{k<n}(i\sigma^z_k)\sigma^-_n$

$\Rightarrow$ Model can be solved with standard MPS techniques

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- Extrapolate to the continuum similar to lattice calculations

Bond dimension: $D$

Infinite volume: $N \times ag$

Continuum: $\frac{1}{(ag)^2}$

# Spectral properties at $\theta = 0$

T. M. R. Byrnes, P. Sriganesh, R. J. Bursill, C. J. Hamer, Phys. Rev. D 66, 013002 (2002)
M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, JHEP 2013, 158 (2013)
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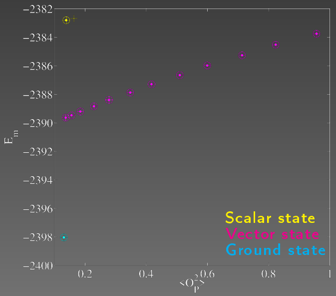
  - DMRG
  - Integrating out the gauge field and using MPS with OBC
  - Truncating the gauge field and using uniform MPS





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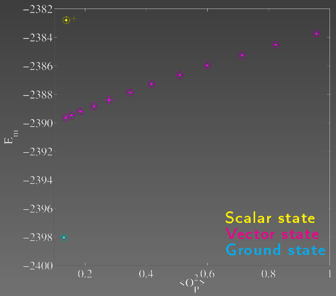


Vector mass

| m/g | Byrnes et al. | Banuls et al. | Buyens et al. |
|---|---|---|---|
| 0 | 0.5641859 | 0.56414(26) | 0.56418(2) |
| 0.125 | 0.53950(7) | 0.53946(20) | 0.539491(8) |
| 0.25 | 0.51918(5) | 0.51915(14) | 0.51917(2) |
| 0.5 | 0.48747(2) | 0.48748(6) | 0.487473(7) |

Scalar mass

| m/g | SCE | Banuls et al. | Buyens et al. |
|---|---|---|---|
| 0 | 1.128379 | 1.1283(10) | - |
| 0.125 | 1.22(2) | 1.221(2) | 1.122(4) |
| 0.25 | 1.24(3) | 1.239(6) | 1.2282(4) |
| 0.5 | 1.20(3) | 1.231(5) | 1.2004(1) |


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# Behavior at $\theta \neq 0$

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B. Buyens, S. Montangero, J. Haegeman, F. Verstraete, K. Van Acoleyen, Phys. Rev. D 95, 094509 (2017)
L. Funcke, K. Jansen, SK, arXiv:1908.00551

# The Schwinger model

Continuum predictions for the behavior

- Small $m/g$: mass perturbation theory
  - Energy density in units of the coupling

$$\frac{\mathcal{E}_0(m,\theta)}{g^2} = c_1 \frac{m}{g}\cos(\theta) + \frac{\pi c_1^2}{4}\left(\frac{m}{g}\right)^2 (c_2 \cos(2\theta) + c_3)$$

C. Adam, Ann. Phys. 259, 1 (1997)

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  - Electric field in units of the coupling

$$\frac{\mathcal{F}(m,\theta)}{g} = 2\pi \times \frac{\partial}{\partial\theta}\frac{\mathcal{E}_0(\theta,m)}{g^2}$$

  - Topological susceptibility in units of the coupling

$$\frac{\chi_{\mathrm{top}}(m,\theta)}{g} = -\frac{\partial^2}{\partial\theta^2}\frac{\mathcal{E}_0(\theta,m)}{g^2} = -\frac{1}{2\pi}\frac{\partial}{\partial\theta}\frac{\mathcal{F}(\theta,m)}{g}$$

- For $m/g = 0$ physics is independent of $\theta$
- Simultaneous shift $m \to -m$, $\theta \to \theta + \pi$ leaves quantities invariant

C. Adam, Ann. Phys. 259, 1 (1997)

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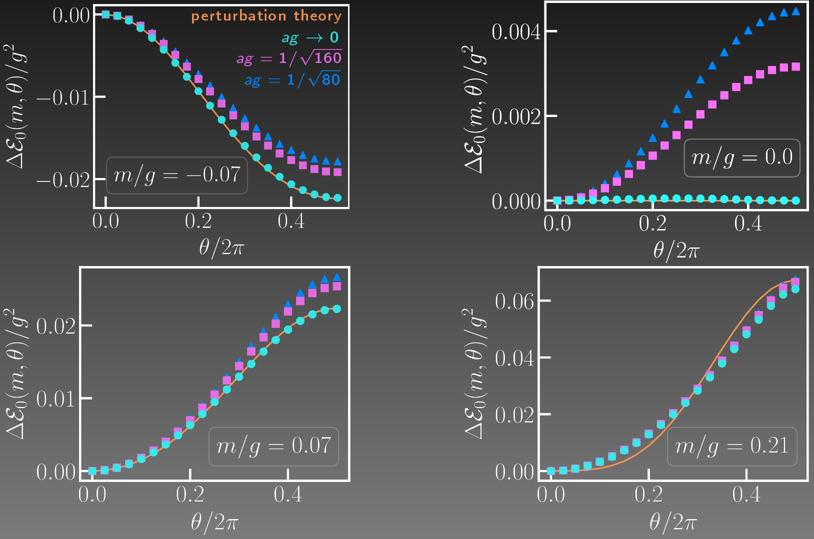
## Results for the energy density

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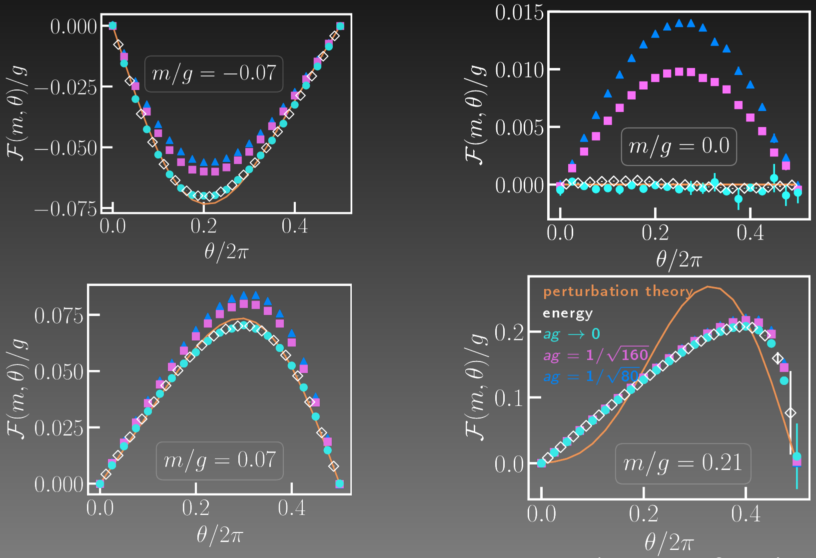


L. Funcke, K. Jansen, SK, arXiv:1908.00551

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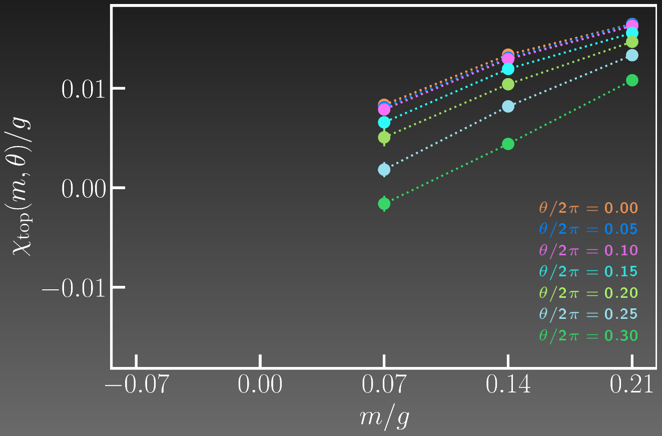
## Results for the topological susceptibility



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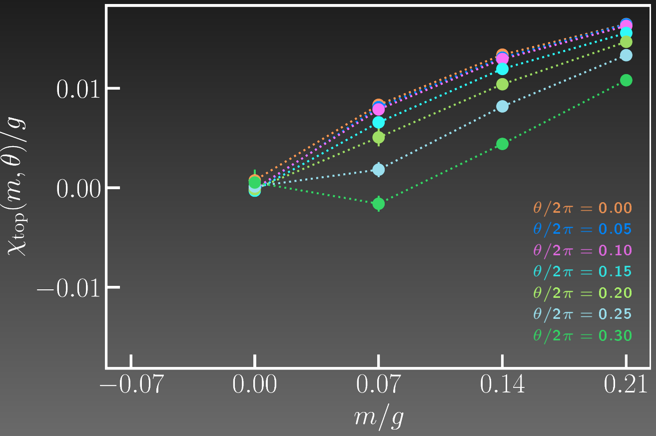


$\Rightarrow$ For $m/g = 0$ physics is independent of $\theta$, CP symmetry is restored

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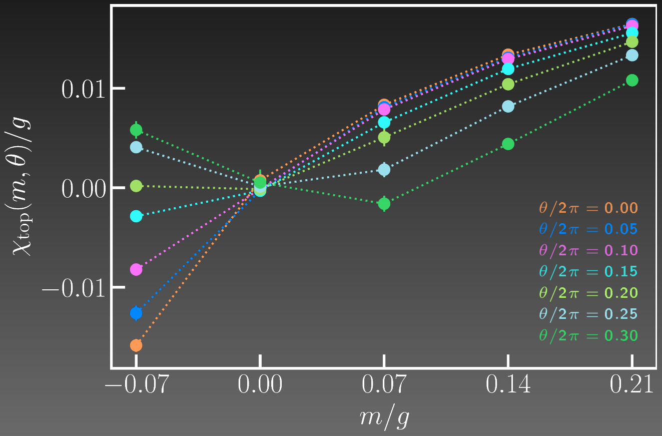


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# Finite temperature

M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, H. Saito Phys. Rev. D 92, 034519 (2015)
M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, H. Saito Phys. Rev. D 93, 094512 (2016)
B. Buyens, F. Verstraete, K. Van Acoleyen, Phys. Rev. D 94, 085018 (2016)

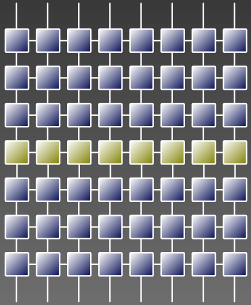
# The Schwinger model

## Nonzero temperature with MPS

- Mixed state at inverse temperature $\beta = 1/T$

$$\rho(\beta) \propto \exp\left(-\beta H\right) = \exp\left(-\frac{\beta}{2} H\right) \mathbb{1} \exp\left(-\frac{\beta}{2} H\right)$$

$$\rho(\beta) =$$



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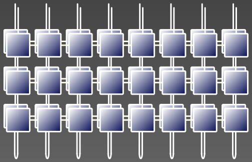
$\rho(\beta) =$ 

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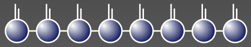
$\rho(\beta) =$ 

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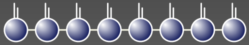
$\rho(\beta) =$ 

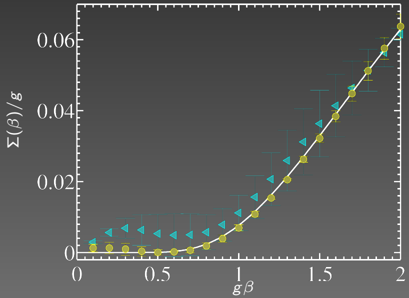
- Can be computed with imaginary time evolution

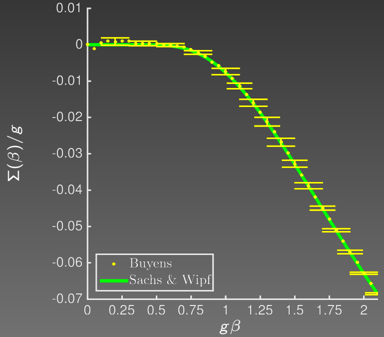
# The Schwinger model

## Nonzero temperature with MPS

- Order parameter for chiral symmetry breaking

$$\frac{\Sigma}{g} = \frac{\langle\overline{\psi}\psi\rangle}{g} \equiv \frac{1}{Nag}\sum_n (-1)^n \frac{1+\sigma_n^z}{2}$$




$\Rightarrow$ Chiral symmetry is restored for large temperatures


M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, H. Saito Phys. Rev. D 92, 034519 (2015)
B. Buyens, F. Verstraete, K. Van Acoleyen, Phys. Rev. D 94, 085018 (2016)

# Phase structure for two flavors at finite density

M.C. Bañuls, K. Cichy, J.I. Cirac, K. Jansen, SK, Phys. Rev. Lett. 118, 071601 (2017)

# The multiflavor Schwinger model

## Lattice Hamiltonian formulation

- Kogut-Susskind staggered fermions in temporal gauge $A^0 = 0$

$$H = -\frac{i}{2a}\sum_{n=1}^{N-1}\sum_{f=1}^{F}\left(\phi^\dagger_{n,f}\, e^{i\alpha_n}\, \phi_{n+1,f} - \text{h.c}\right) + \sum_{n=1}^{N}\sum_{f=1}^{F}\left((-1)^n m_f + \kappa_f\right)\phi^\dagger_{n,f}\phi_{n,f} + \frac{ag^2}{2}\sum_{n=1}^{N-1} L_n^2$$

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kinetic part + coupling to gauge field — mass term — chemical potential — electric energy

$$\{\phi^\dagger_{n,f}, \phi_{n',f'}\} = \delta_{nn'}\delta_{ff'}, \qquad \alpha_n \in [0, 2\pi], \qquad [\alpha_n, L_{n'}] = i\delta_{nn'}$$

- Gauss law

$$L_n - L_{n-1} = Q_n = \sum_{f=1}^{F}\left(\phi^\dagger_{n,f}\phi_{n,f} - \frac{1}{2}\left(1 - (-1)^n\right)\right)$$

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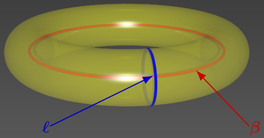
- Use OBC and Gauss law to integrate out the gauge field

$$l_0 + Q_1 \quad + \quad Q_2 \quad + \quad Q_3 = l_3$$

# The multiflavor Schwinger model

## Previous analytic work

- Phase structure for $m/g = 0$ on a torus in the continuum

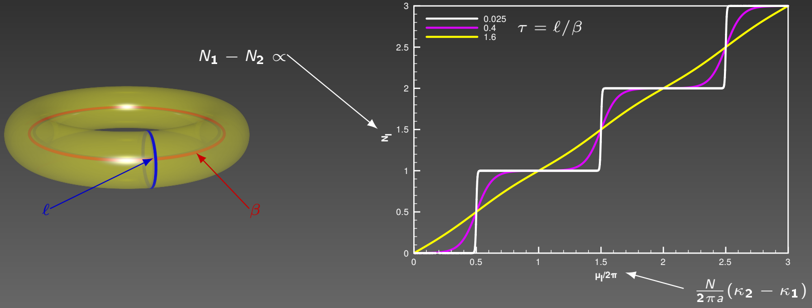



R. Narayanan, Phys. Rev. D 86, 125008 (2012)
R. Lohmayer, R. Narayanan, Phys. Rev. D 88, 105030 (2013)

# The multiflavor Schwinger model

## Previous analytic work

- Phase structure for $m/g = 0$ on a torus in the continuum
- Result for the two-flavor case




- Jumps in $N_1 - N_2$ correspond to first oder phase transitions


R. Narayanan, Phys. Rev. D 86, 125008 (2012)
R. Lohmayer, R. Narayanan, Phys. Rev. D 88, 105030 (2013)

# The multiflavor Schwinger model

## MPS approach to the multiflavor Schwinger model

- Focus on the two-flavor case at zero temperature
- Fix $\kappa_1 = 0$ and vary $\kappa_2 \Rightarrow$ **sign problem for Monte Carlo**
- Use MPS with open boundary conditions
- Fixed physical volume

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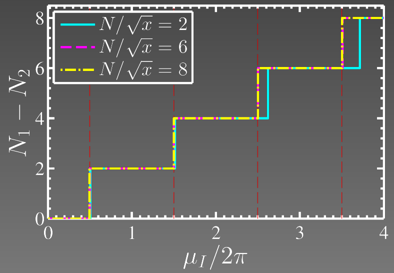

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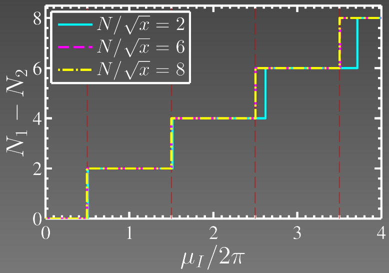
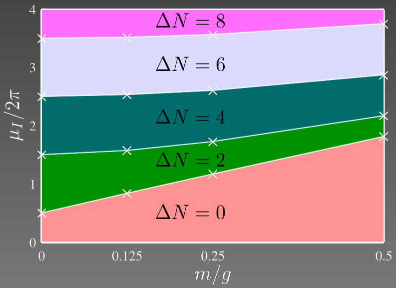



### Nonvanishing fermion mass



# Out-of-equilibrium phenomena

B. Buyens, J. Haegeman, K. Van Acoleyen, H. Verschelde, F. Verstraete, Phys. Rev. Lett 113, 091601 (2014)
T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, Phys. Rev. X 6, 011023 (2016)
B. Buyens, J. Haegeman, F. Hebenstreit, F. Verstraete, K. Van Acoleyen, Phys. Rev. D 96, 114501 (2017)
G. Magnifico, M. Dalmonte, P. Facchi, S. Pascazio, F. V. Pepe, E. Ercolessi, arXiv:1909.04821
Y.-T. Kang, C.-Y. Lo, S. Yin, P. Chen, arXiv:1910.01320

# The Schwinger model

## String breaking in the Schwinger model

- Insert a pair of charges in the bare vacuum
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- From a certain length on it is energetically favorable to break the string and generate particle-antiparticle pairs

T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, Phys. Rev. X 6, 011023 (2016)

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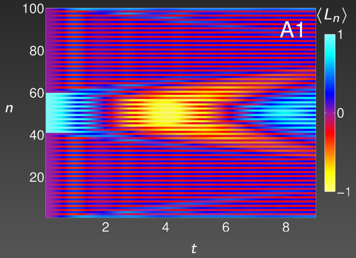



T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, Phys. Rev. X 6, 011023 (2016)

# SU(2) Lattice Gauge Theory

## Lattice Hamiltonian formulation

- Kogut-Susskind staggered fermions in temporal gauge $A^0 = 0$

$$H = \varepsilon \sum_{n=1}^{N-1} \left( \phi_n^\dagger \, U_n \, \phi_{n+1} + \text{H.c.} \right) + m \sum_{n=1}^{N} (-1)^n \phi_n^\dagger \phi_n + \frac{g^2}{2} \sum_{n=1}^{N-1} \boldsymbol{L}_n^2$$

kinetic part + coupling to gauge field

staggered mass term

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J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975)

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- Suitable basis

$$|n_r, n_g\rangle \otimes |jmm'\rangle \otimes |n_r, n_g\rangle \otimes \dots$$

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- Suitable basis for the gauge invariant subsapce

$$|n\rangle \otimes |j\rangle \otimes |n\rangle \otimes \ldots$$

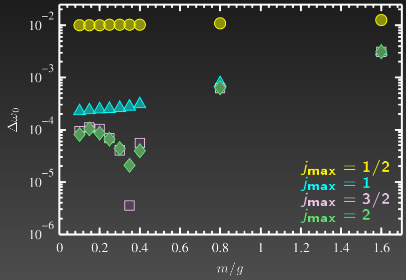
M. C.Bañuls, K. Jansen, SK, Phys. Rev. X 7, 041046 (2017)
J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975)

# Spectral properties

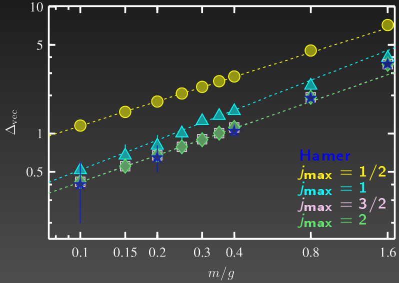
M. C.Bañuls, K. Jansen, SK, Phys. Rev. X 7, 041046 (2017)

# SU(2) Lattice Gauge Theory

## Ground state




## Vector mass gap




C. J. Hamer Nucl. Phys. B 195, 503 (1982)

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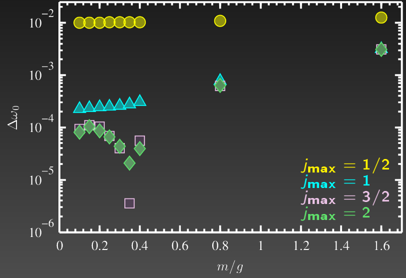


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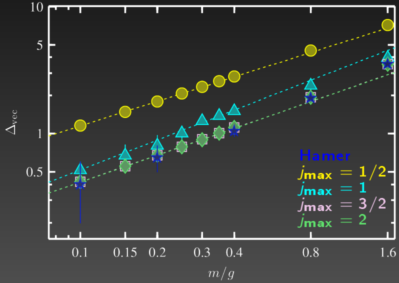


## Critical exponent for the vector mass gap

- Scaling of $\Delta_{vec}$ for $m/g \to 0$:

$$\Delta_{vec} \propto \left(\frac{m}{g}\right)^{\gamma}$$

- Large $N_c$: $\gamma = 2/3$

| $j_{max}$ | Exponent |
|---|---|
| 1/2 | $0.639(43)_{stat}(5)_{sys}$ |
| 1 | $0.781(93)_{stat}(65)_{sys}$ |
| 3/2 | $0.700(29)_{stat}(11)_{sys}$ |
| 2 | $0.700(29)_{stat}(12)_{sys}$ |

C. J. Hamer Nucl. Phys. B 195, 503 (1982)

# Entanglement Entropy

M. C.Bañuls, K. Jansen, SK, Phys. Rev. X 7, 041046 (2017)

# SU(2) Lattice Gauge Theory

## Entanglement entropy

- Gauge constraints are not purely local

$\Rightarrow$ Not all entropy is physical

P. Calabrese, J. Cardy, J. Stat. Mech. 2004, P06002 (2004)
S. Ghosh, R. M. Soni, S. P. Trivedi, J. High Energy Phys. 2015, 69 (2015)
R.M. Soni, S. P. Trivedi, J. High Energy Phys. 2016, 136 (2016)
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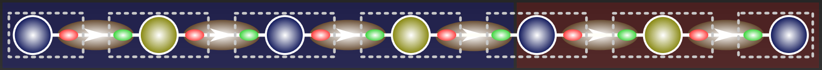

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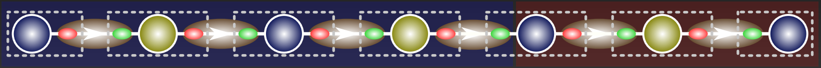

$\Rightarrow$ Not all entropy is physical

- The reduced density matrix is of the form $\rho = \oplus_j (\bar{\rho}_j \otimes \mathbb{1}_j)$

$$S(\rho) = \underbrace{-\sum_j p_j \log_2(p_j)}_{\text{classical part}} + \underbrace{\sum_j p_j \log_2(2j+1)}_{\text{representation part}} + \underbrace{\sum_j p_j S(\bar{\rho}_j)}_{\text{physical part}}$$

- Towards the continuum limit

$$S(\rho) = \frac{c}{6} \log_2 \left( \frac{\xi}{a} \right)$$

P. Calabrese, J. Cardy, J. Stat. Mech. 2004, P06002 (2004)
S. Ghosh, R. M. Soni, S. P. Trivedi, J. High Energy Phys. 2015, 69 (2015)
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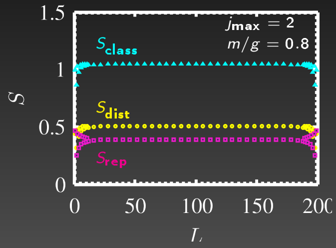
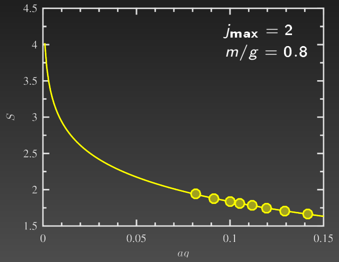
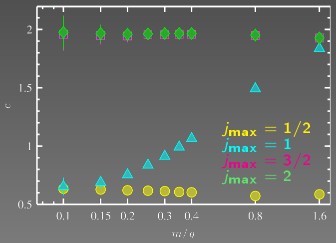
# SU(2) Lattice Gauge Theory

## Entanglement entropy

- MPS allow for accessing the different parts of the entropy



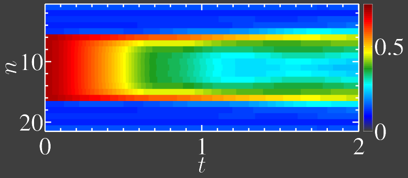




M. C.Bañuls, K. Jansen, SK, Phys. Rev. X 7, 041046 (2017)

## More results on non-Abelian LGT in 1+1 dimension

# Non-Abelian Lattice Gauge Models

## String breaking in a SU(2) LGT

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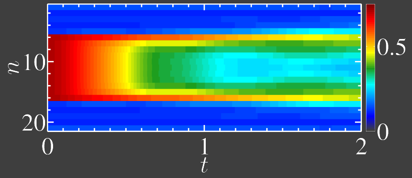
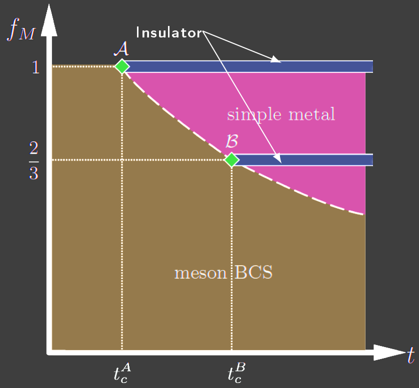



## Phase structure of a SU(2) LGT




M. C.Bañuls, K. Jansen, SK, J. High Energy Phys. 2015, 130 (2015)
P. Silvi, E. Rico, M. Dalmonte, F. Tschirsich, S. Montangero, Quantum 1, 9 (2017)

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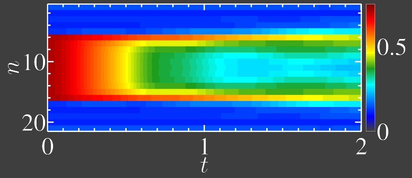
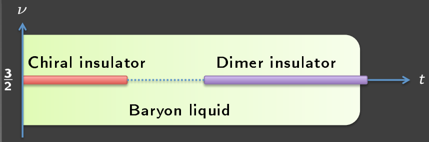
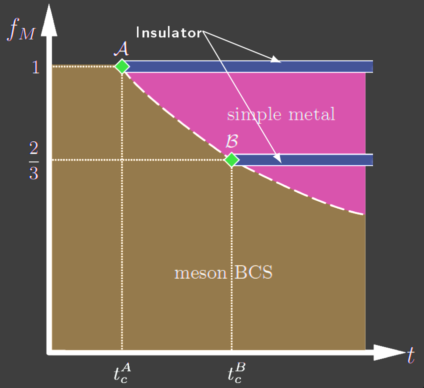



## Phase structure of a SU(3) LGT




## Phase structure of a SU(2) LGT





M. C.Bañuls, K. Jansen, SK, J. High Energy Phys. 2015, 130 (2015)
P. Silvi, E. Rico, M. Dalmonte, F. Tschirsich, S. Montangero, Quantum 1, 9 (2017)
P. Silvi, Y. Sauer, F. Tschirsich, S. Montangero, Phys. Rev. D 100, 074512 (2019)

# The O(3) rotor model

## Continuum formulation

- Euclidean time Lagrangian of the model

$$\mathcal{L}_{\mathsf{O(3)}} = \frac{1}{2g_0^2}\partial_\nu \mathbf{n} \partial_\nu \mathbf{n}, \qquad \mathbf{n} \in \mathbb{R}^3, \qquad \mathbf{n}\cdot\mathbf{n} = 1$$

$\Rightarrow$ Nonlinearity because of the constraint

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$\Rightarrow$ Nonlinearity because of the constraint

- Focus on two (1+1) dimension: $\nu = 0, 1$
  - Asymptotic freedom
  - Dynamically generated mass gap
  - Nontrivial topology and instantons

*"The $CP^{N-1}$ system does its best to imitate QCD."*

– A. Actor, Fortschr. Phys., 33: 333-374 (1985)

# The O(3) rotor model

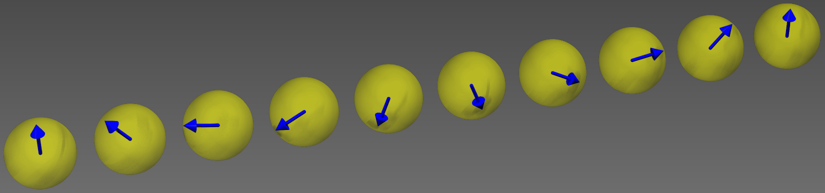
## Lattice discretization

- Dimensionless Hamiltonian on a lattice with spacing $a$

$$aH = \frac{1}{2\beta}\sum_{k=1}^{N}\mathbf{L}_k^2 - \beta\sum_{k=1}^{N-1}\mathbf{n}_k\mathbf{n}_{k+1}$$

$$[L^\alpha, L^\beta] = i\varepsilon^{\alpha\beta\gamma}L^\gamma, \qquad [L^\alpha, n^\beta] = i\varepsilon^{\alpha\beta\gamma}n^\gamma, \qquad [n^\alpha, n^\beta] = 0$$

- Hamiltonian describes chain of coupled quantum rotors



- Continuum limit: $\beta \to \infty$ (thanks to asymptotic freedom)

# Spectral properties

F. Bruckmann, K. Jansen, SK, Phys. Rev. D 99, 074501 (2019)

## Spectral properties

### Numerical approach

- Suitable basis: angular momentum eigenstates $\otimes|l_k m_k\rangle_{k=1}^{N}$
- Angular momentum of each rotor is unbounded

$\Rightarrow$ Local Hilbert spaces are infinite dimensional

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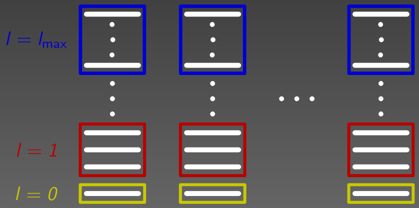
## Numerical approach

- Suitable basis: angular momentum eigenstates $\otimes|l_k m_k\rangle_{k=1}^{N}$
- Angular momentum of each rotor is unbounded

$\Rightarrow$ Local Hilbert spaces are infinite dimensional

- Truncate maximum angular momentum at each site

$$l \leq l_{\max} \qquad \forall k$$




$\Rightarrow$ Local Hilbert spaces of dimension $(l_{\max}+1)^2$

- Use MPS with OBC to solve the model

# Spectral properties

## Mass gap

- Asymptotic scaling

$$am = \frac{8}{e}\, a\Lambda_{\overline{\mathrm{MS}}} = 64\, a\Lambda_L = 128\pi\beta \exp(-2\pi\beta)$$

Hasenfratz, Maggiore, Niedermayer, Phys. Lett. B, 245 522 (1990)
Shigemitsu, Kogut, Nucl. Phys. B, 190, 365 (1981)

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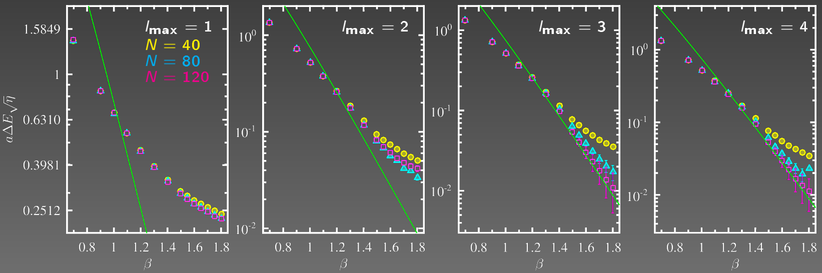
- Numerical data




$\Rightarrow$ Good agreement with the analytical prediction

Hasenfratz, Maggiore, Niedermayer, Phys. Lett. B, **245** 522 (1990)
Shigemitsu, Kogut, Nucl. Phys. B, **190**, 365 (1981)

# Numerical results

## Entanglement entropy

- As we approach $\beta \to 0$ the mass gap closes
- The correlation length in lattice units $\xi/a = 1/am$ diverges

$$S = \frac{c}{6}\log\frac{\xi}{a} + \bar{k} = \frac{c}{6}\left(2\pi\beta - \log\beta\right) + k$$

Calabrese, Cardy, J. Stat. Mech., 2004, P06002 (2004)

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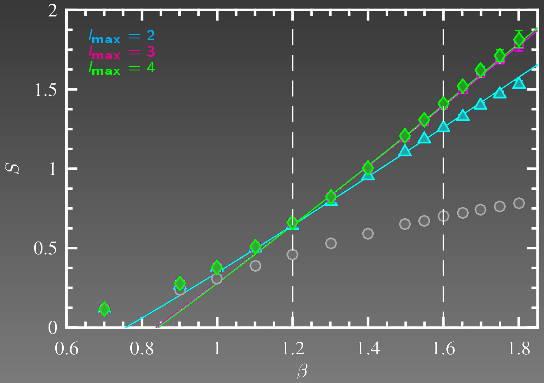


| $l_{\mathrm{max}}$ | $c$ |
|---|---|
| 2 | $1.66 \pm 0.19$ |
| 3 | $2.01 \pm 0.12$ |
| 4 | $2.04 \pm 0.14$ |

Calabrese, Cardy, J. Stat. Mech., 2004, P06002 (2004)

# Phase structure at nonvanishing chemical potential

# Phase structure at nonvanishing chemical potential

## Adding a chemical potential

- Conventional Monte Carlo approach: **sign problem**
- Lattice Hamiltonian with chemical potential

$$aH = \frac{1}{2\beta}\sum_{k=1}^{N}\mathbf{L}_k^2 - \beta\sum_{k=1}^{N-1}\mathbf{n}_k\mathbf{n}_{k+1} - a\mu Q$$

$$Q = \sum_{k=1}^{N} L_k^z, \qquad [H, Q] = 0$$

- Hamiltonian is block diagonal, inside a block with charge eigenvalue $q$

$$aH|_q = -a\mu q\mathbb{1} + aW_{\text{aux}}|_q.$$

- Ground-state energy inside a block with charge $q$

$$aE_{0,q}(\mu) = -a\mu q + aE_0[aW_{\text{aux}}|_q]$$

F. Bruckmann, C. Gattringer, T. Kloiber, T. Sulejmanpasic, Phys. Rev. D, **94**, 114503 (2016)

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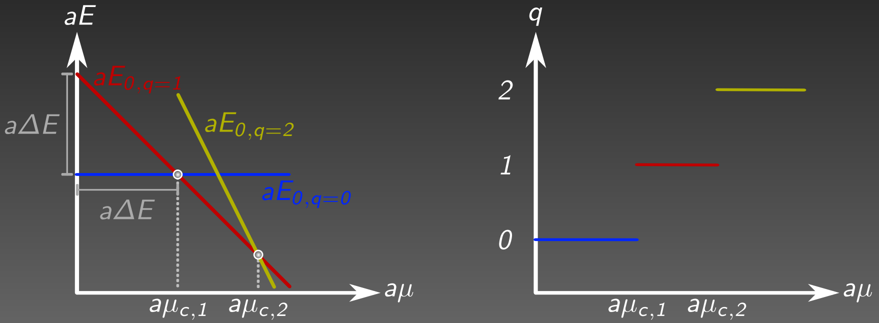
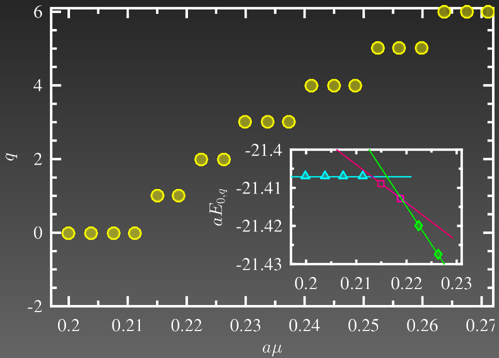



- For $a\mu = 0$:
  - Ground state is in sector $q = 0$
  - First excited states: massive particles form a triplet with $q = 0, \pm 1$

# Phase structure at nonvanishing chemical potential

## Phase structure at nonvanishing chemical potential

- Numerical data for $l_{\mathrm{max}} = 4$, $\beta = 1.2$, $N = 80$ and $D = 200$

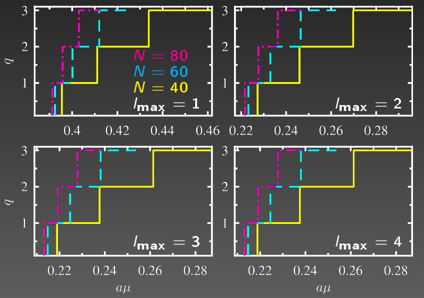



$\Rightarrow$ Excellent agreement with the analytical prediction

- Intersection of the energy levels allows for determining the transition points precisely with small resolution in $a\mu$

## Phase structure at nonvanishing chemical potential

- Numerical data for $\beta = 1.2$ and various $N$, $l_{\max}$



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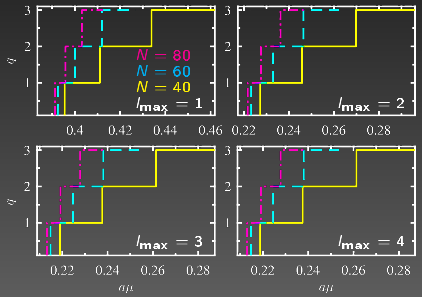


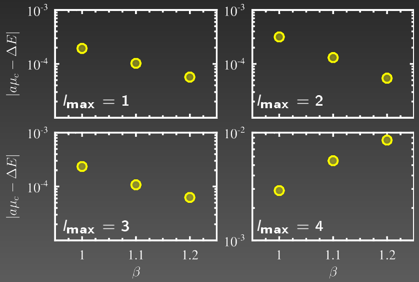


- Noticeable truncation effects only for $l_{\max} = 1$
- $a\mu_c$ at first transition is in excellent agreement with the gap

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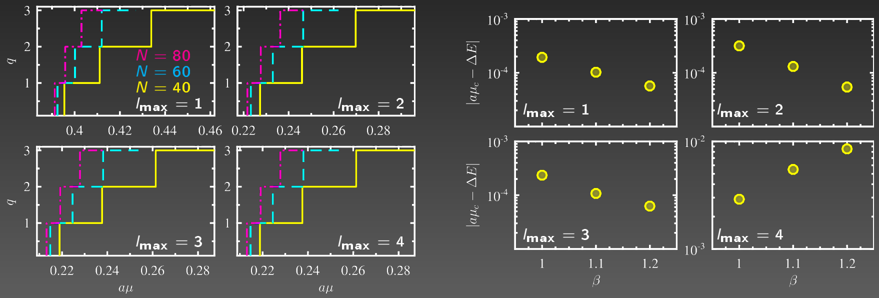


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- $a\mu_c$ at first transition is in excellent agreement with the gap

$\Rightarrow$ Example of overcoming the sign problem

# Summary

## To summarize

- Feasibility of addressing Lattice Field Theories with TNS has been demonstrated
  - Spectral properties
  - Thermal states
  - Out-of-equilibrium dynamics
- Good numerical precision attainable, possible to extract continuum data

# Summary

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- Feasibility of addressing Lattice Field Theories with TNS has been demonstrated
  - Spectral properties
  - Thermal states
  - Out-of-equilibrium dynamics
- Good numerical precision attainable, possible to extract continuum data

## Outlook

- Many more interesting questions in 1+1 dimension
  - Models with topological $\theta$-term
  - Quantum computing
- Study 2+1 dimensional models

Y. Kuramashi, Y. Yoshimura, J. High Energy Phys. 2019, 23 (2019)
T. Felser, P. Silvi, M. Collura, S. Montangero, arXiv:1911.09693

# A. Hamiltonian lattice formulation for gauge theories

## Gauge Hamiltonian formulation on the lattice

- Kogut-Susskind staggered fermions in temporal gauge $A^0 = 0$

$$H = \varepsilon \sum_{n=1}^{N-1} \left( \phi_n^\dagger U_n \phi_{n+1} + \text{h.c.} \right) + m \sum_{n=1}^{N} (-1)^n \phi_n^\dagger \phi_n + \frac{g^2}{2} \sum_{n=1}^{N-1} \mathbf{L}_n^2$$

Kinetic part + coupling to gauge field

staggered mass term

(color)-electric energy

J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975)

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Kinetic part + coupling to gauge field

staggered mass term

(color)-electric energy

- Gauss law for physical states $G_n^a |\psi\rangle = 0$

$$G_n^a = L_n^a - R_{n-1}^a - \mathcal{Q}_n^a, \quad \mathcal{Q}_n = \boldsymbol{Q}_n + \boldsymbol{q}_n$$

dynamical charge

external charge

J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975)

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Kinetic part + coupling to gauge field — staggered mass term — (color)-electric energy

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## U(1), Schwinger model

- ▷ Single-component fermionic field: $\phi_n$
- ▷ $Q_n = \phi_n^\dagger \phi_n - \frac{1}{2}(1 - (-1)^n)$
- ▷ $q_n \in \mathbb{R}$

## SU(2)

- ▷ Two colors of fermions: $\phi_n^\dagger = (\phi_n^{r,\dagger}, \phi_n^{g,\dagger})$
- ▷ $Q_n^a = \frac{1}{2} \phi_n^\dagger \sigma^a \phi_n$
- ▷ $q_n^a = \frac{1}{2} \sigma^a$

J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975)

# B. Hamiltonian lattice formulation for gauge theories

## Disentangling the gauge field for open boundary conditions

- Transformation disentangling the gauge degrees of freedom

$$\Theta = \overrightarrow{\prod_{k=1}} \exp\left( i\theta_k^a \sum_{m>k} \mathcal{Q}_m^a \right)$$

- Hamiltonian in the rotated frame $H_\Theta = \Theta H \Theta^\dagger$

$$H_\Theta = \varepsilon \sum_n \left( \phi_n^\dagger \phi_{n+1} + \mathrm{H.c.} \right) + m \sum_n (-1)^n \phi_n^\dagger \phi_n + \sum_a \sum_{n,m} \mathcal{Q}_n^a V_{n,m} \mathcal{Q}_m^a$$

- In the sector of vanishing total charge $\sum_n \boldsymbol{\mathcal{Q}}_n = 0$

$$V_{n,m} = -\frac{1}{2} |n - m|$$

- $H_\Theta$ depends only on the fermionic content and is nonlocal

F. Lenz, H.W.L. Naus, M. Thies, Ann. Phys. 233, 317 (1994)
B. Bringoltz, Phys. Rev. D 79, 105021 (2009)

# C. Euclidean time lattice Schwinger model

## Continuum formulation

- Euclidian action of the model

$$S = \int d^2x \, \underbrace{\overline{\Psi}\gamma_\mu D^\mu \Psi}_{\text{kinetic energy + coupling to the gauge field}} + \underbrace{m\overline{\Psi}\Psi}_{\text{mass term}} + \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{dynamics of gauge field}}$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \; D_\mu = \partial_\mu + igA_\mu, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

J. Schwinger, Phys. Rev. 128 2425 (1962)

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- Simplest nontrivial gauge theory with matter
- Many similarities with QCD
  - Confinement (of charges)
  - Bound states
  - Chiral symmetry breaking

J. Schwinger, Phys. Rev. 128 2425 (1962)

# C. Euclidean time lattice Schwinger model

## Lattice discretization

- Regular lattice with spacing $a$
- Fermionic fields: sit at the vertices of the lattice

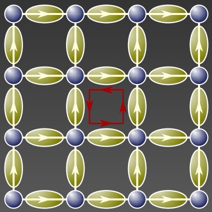
$$\Psi(x) \to \psi_n$$

- Derivative

$$\partial_u \Psi(x) \to \hat{\partial}_u \hat{\psi}_n = \frac{1}{2a}\left(\psi_{n+a\mu} + \psi_{n-a\mu}\right)$$

- Discretized fermionic action

$$S = \sum_n \overline{\psi}_n \left(\gamma_u \hat{\partial}_u + m\right) \psi_n$$



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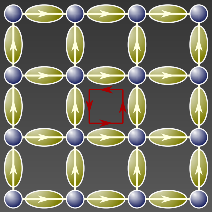

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$$S = \sum_n \overline{\psi}_n \left(\gamma_u \hat{\partial}_u + m\right) \psi_n - \boxed{\frac{r}{2} \sum_n \overline{\psi}_n \hat{\Box} \psi_n}$$

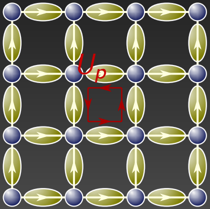
removing the doublers by breaking chiral symmetry

# C. Euclidean time lattice Schwinger model

## Lattice discretization

- Gauge invariance in the continuum
  $\overline{\Psi}(x)U(x,y)\Psi(y)$
  where

$$U(x,y) = \exp\left(ig\int_x^y dz_\mu A_\mu(z)\right) \in \mathrm{U}(1)$$




- Plaquette term
  $U_p = U_\mu(n)U_\nu(n+\mu)U_\mu^\dagger(n+\nu)U_\nu^\dagger(n)$
- Euclidean time action

$$S = -\frac{1}{2}\sum_{n,\mu}\left[\overline{\psi}_n(r-\gamma_\mu)U_\mu(n)\psi_{n+\mu} + \overline{\psi}_{n+\mu}(r+\gamma_\mu)U_\mu^\dagger(n)\psi_n\right]$$

$$(M+4r)\sum_n \overline{\psi}_n\psi_n + \frac{1}{g^2}\sum_p(1-\mathrm{Re}\,U_p)$$

# C. Euclidean time lattice Schwinger model

## Lattice Field Theory

- Lattice action

$$S_{\text{matter}} = \sum_{ij} \overline{\psi}_i K_{ij}[U] \psi_j$$

- Path integral in Euclidean time

$$Z = \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, \mathcal{D}U \exp\Big( -S_{\text{gauge}}[U] - S_{\text{matter}}\left[\psi, \overline{\psi}, U\right] \Big)$$

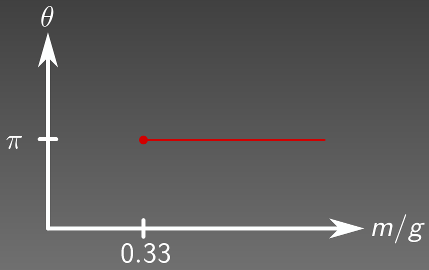
- Fermionic degrees of freedom can be integrated out analytically

$$Z = \int \mathcal{D}U \exp\Big( -S_{\text{gauge}}[U] \Big) \det\big( K[U] \big)$$

# C. Lattice Schwinger model

## Continuum predictions for the behavior

- Conventional Monte Carlo approach suffers from the sign problem for $\theta \neq 0$
- Prediction for the phase structure by Coleman
  - Nontrivial topological vacuum structure gives rise to $\theta$
  - Phase structure at $\theta = \pi$





S. R. Coleman, Annals Phys. 101, 239 (1976)
T. M. R. Byrnes, P. Sriganesh, R. J. Bursill, C. J. Hamer, Phys. Rev. D 66, 013002 (2002)